Review of paper “Some solutions to a third-order quaternion tensor equation”

The paper focuses on the study of a third-order quaternion tensor equation. By utilizing the so-called "Qt multiplication operation", the authors establish conditions and obtain various solutions, including the general, least-squares, minimum-norm, and minimum-norm least-squares solutions for the tensor equation $A \ast X = B$. A numerical example is provided to illustrate the concepts discussed.

Based on my understanding and the literature review I conducted, I believe that the solution to the linear equation is innovative and remains unresolved to this day.

However, certain modifications are required to enhance the structure of the paper and increase its chances of acceptance. Furthermore, the contribution of the paper needs to be clarified. The following suggestions are proposed to address these concerns:

1. The introduction lacks clarity, as there is a disconnect between the first and second paragraphs. The second paragraph should include relevant references to existing solutions for a third-order tensor equation involving complex numbers. Moreover, the motivation and contribution of the paper need to be clearly articulated to provide a comprehensive overview. A potential suggestion for enhancing the introduction could be as follows:

   a. Firstly, I suggest introducing the concept of a third-order tensor complex numbers entries, along with its linear equation and applications. The authors can refer to the following sources for additional information:


b. Second, introduce the concept of third-order quaternion tensor using the following references:


c. Third, provide the motivation behind extending the research carried out in [Ref 2] and [Ref 6], which focused on linear equations and least-squares solutions for third-order tensors of complex numbers, to the domain of third-order quaternion tensors. Additionally, to present the contribution, which entails solving a third-order quaternion tensor equation and deriving its corresponding least-squares solution.

d. Finally, the authors should include a discussion on prior works in the introduction, outlining the differences between these references and the proposed method in the paper. Some relevant references to consider are as follows:


2. In Section 2; Preliminary Results, it would be beneficial to include references that serve as the foundation for the notation employed in this paper.

3. Definition of a third-order tensor needs to be modified, because $A = (a_{(i_1)}a_{(i_2)}a_{(i_3)})$, is incorrect. The correct definition is $A = (a_{(i_1i_2i_3)})$, i.e., just write the letter "a" once.
4. Definition 2 is not clear. Please, use the definition presented in Equation (1) in reference [Ref 6].

5. Equation (2.3) doesn't fit in Section 2: Preliminary Results, because Equation (2.3) is related to tensors with complex entries, and Section 2 is related to tensors with quaternions entries. I propose incorporating equation (2.3) into the proof of Proposition 2, where it is utilized.

6. Before Proposition 1, add the following sentence to the end of the paragraph: "Since the proofs follow a similar approach as with matrices, we omit them here."

7. The proof of Proposition 2 references equation (2.1), which is related to tensors with quaternion entries, whereas Proposition 2 deals with tensors with complex entries. To clarify this situation, please refer to [Ref 4], Section 2.2, where Equation (2.1) is presented specifically for tensors with complex entries.

8. In Section 3: Generalized Inverses, after Definition 3, the authors state the following: "In [30], the authors derived $A^\dagger$ based on the SVD decomposition of $A$ and the $Qt$-product between the two third-order quaternion tensors," and present Lemma 2. However, the definition of the pseudoinverse of third-order quaternion tensors is not provided in [30]. To address this, please clarify the situation or present the proof for Lemma 2 and Proposition 4.

9. Definition 4 pertains to the four Penrose equations. However, the authors write $A_{i,j,...,k}$, which is confusing. It would be better to use $A_{1,2,3,4}$ instead. Additionally, please adjust Equation (3.1) to include a number label at the beginning of each equation to enumerate them individually.

10. In the proof of Theorem 3, the authors utilize the null/zero quaternion tensor; however, this term is not defined in the paper. Please provide a definition for the null/zero quaternion tensor.

11. The Frobenius norm of a quaternion tensor should be included in Section 2: Preliminary Results, along with equations (4.6) and (4.7), instead of Section 4.

12. In Theorem 4, specify that $\text{Str}()$ represents the trace of a matrix.

13. Equation (4.11) should not simply use the word "min" as it is not mathematically correct. Please correct this equation accordingly.

14. In Section 5: Numerical Example, the presented numerical example is too simple. To enhance the section, please include a real-world application example using quaternion tensors, such as denoising an image similar to the examples presented in [Ref 1], [Ref 2], [Ref 3], or [Ref 4]. Also, I suggest providing a link where the files with computational implementation are included. For example, authors can use GitHub to store the code for the numerical examples presented in the document.