Study of the fuzzy $q$–spiral-like functions associated with generalized linear operator

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Abstract: Nowadays, the subclasses of analytic functions in terms of fuzzy subsets are studied by various scholars and some of these concepts are extended by using the $q$–theory of functions. In this inspiration, we introduce certain subclasses of analytic function by using the notion of fuzzy subsets along with the idea of $q$–calculus. This article presents the $q$–extensions of the fuzzy spiral-like functions of complex order. We generalize this class by using the $q$–analogues of the Ruscheweyh derivative and Srivastava-Attiya operators. Various interesting properties are examined for the newly defined subclasses. Also, some previously investigated results are deduced as the corollaries of our main results.

Keywords: Analytic functions; $q$–spiral-like; fuzzy $q$–starlike functions; fuzzy $q$–convex functions; $q$–Ruscheweyh derivative operator; $q$–Srivastava-Attiya operator

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1. Introduction

The article written by Lotfi A. Zadeh and published in 1965 [1] serves as the foundation for fuzzy sets theory. In response to the numerous attempts by researchers to connect this theory with various areas of mathematics, the connection between fuzzy sets theory and the area of complex analysis that
studies analytic functions by virtue of their geometric properties was established in 2011 [2]. The authors in [3, 4] introduced the concept of differential subordination. G.I. Oros and Gh. Oros [2] researched the idea of fuzzy subordination in 2011, and the same authors [5] introduced the idea of fuzzy differential subordination in 2012. The history of the idea of a fuzzy set and its ties to various scientific and technical fields are nicely reviewed in the 2017 paper [6], including references to the findings made up to that point to the fuzzy differential subordination concept. The first findings confirmed the direction of the research, adapting the traditional theory of differential subordination to the novel features of fuzzy differential subordination and providing techniques for investigating dominants and best dominants of fuzzy differential subordinations [7], without which the research could not have continued. Following that, the particular form of Briot-Bouquet fuzzy differential subordinations was examined [8]. The scholar in [9] adopted the concept and begun to look into the new findings on fuzzy differential subordinations. In this sequel, fuzzy differential subordinations were associated with different operators [10, 11] giving a new direction to the study. Numerous studies [12, 13, 14] carried out the investigations employing certain linear operators. Furthermore, the work of several scholars about the fuzzy differential subordination is referred to the readers, for example, see [15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. The concept of fuzzy differential subordination is the first attempt to incorporate the idea of a fuzzy set into research pertaining to geometric theory of analytic functions. Recently, the authors [25, 26, 27] linked the notion of a fuzzy subsets with the concepts of differential subordination. The classical results of univalent functions with concept of differential subordination are generalised in this current paper to include quantum (q-)extensions of univalent functions associated with fuzzy differential subordination.

Let $\Gamma (\Pi )$ denotes the class of analytic functions $h(\nu )$ in $\Pi = \{ \nu : |\nu| < 1 \}$. The functions $h \in \Gamma (\Pi )$ of the form
\[
h(\nu ) = \nu + a_{\eta+1} \nu^{\eta+1} + a_{\eta+2} \nu^{\eta+2} + \ldots, \quad (\nu \in \Pi ),
\]
form the class denoted by $\mathfrak{A}_\eta$. We note that $\mathfrak{A}_1 = \mathfrak{A}$; the class of normalized analytic functions in $\Pi$. Let $ST$ and $CV$ denote the subclasses of $\mathfrak{A}$ of starlike and convex univalent functions, respectively. Here, we provide an overview of some important fundamental ideas connected to our work.

**Definition 1.1.** [28] A function $F$ is said to be fuzzy subset on $\mathfrak{A} \neq \emptyset$, if it maps from $\mathfrak{A}$ to $[0, 1]$.

In other words, fuzzy subset is defined as;

**Definition 1.2.** [28] A pair $(U, F_U)$ is said to be a fuzzy subset on $\mathfrak{A}$, where $U = \{ x \in Y : 0 < F_U(x) \leq 1 \} = \sup (U, F_U)$ is the support of fuzzy set $(U, F_U)$ and $F_U : \mathfrak{A} \rightarrow [0, 1]$ is the membership function of the fuzzy set $(U, F_U)$.

**Definition 1.3.** [28] Let $(U_1, F_{U_1})$ and $(U_2, F_{U_2})$ be two subsets of $\mathfrak{A}$. Then, $(U_1, F_{U_1}) \subseteq (U_2, F_{U_2})$ if and only if $F_{U_1}(t) \leq F_{U_2}(t)$, $t \in \mathfrak{A}$, whereas, $(U_1, F_{U_1})$ and $(U_2, F_{U_2})$ of $\mathfrak{A}$ are equal if and only if $U_1 = U_2$.

Miller and Mocanu [29] introduced that, let the analytic function $h$ is subordinate to the analytic function $g$ (written as $h \prec g$). Then, $h(\nu) = g(w(\nu))$, where $w(\nu)$ is a Schwartz function in $\Pi$.

The generalization of subordination technique of analytic functions in terms of fuzzy notion was defined by Oros and Oros [5] as the following.
Let the analytic function $h$ be fuzzy subordinate to the analytic function $g$ (written as $h \preceq_R g$). Then,

$$h(v_0) = g(v_0) \text{ and } \mathbb{F}(h(v)) \leq \mathbb{F}(g(v)), \quad v \in \mathcal{R},$$

where $\mathcal{R} \subset \mathbb{C}$ and $v_0$ be a fixed point in $\mathcal{R}$.

**Remark 1.1.** If $\mathcal{R} = \Pi$ in the above definition, then the fuzzy subordination is equivalent to the classical subordination.

For $0 < q < 1$, the operator $\nabla_q$ defined by

$$\nabla_q h(v) = \frac{h(v) - h(qv)}{(1 - q)v}; \quad q \neq 1, \quad v \neq 0,$$

is called the $q$–difference operator. It was introduced by Jackson [30] and it is clear that $\lim_{q \to 1^-} \nabla_q h(v) = h'(v)$, where $h'(v)$ denotes the derivative of the function.

For $j \in \mathbb{N} = \{1, 2, 3, \ldots\}$, we have

$$\nabla_q \left\{ \sum_{j=1}^{\infty} a_j \nu^j \right\} = \sum_{j=1}^{\infty} [\nabla_q a_j] \nu^{j-1},$$

where

$$[\nabla_q a_j] = \frac{1 - q^j}{1 - q} = 1 + q + q^2 + \ldots + q^{j-1}.$$

We have the following rules of $\nabla_q$, we refer to [31, 32].

(i) $\nabla_q (a b_1 \nu) = a \nabla_q b_1 \nu + b \nabla_q a \nu$.

(ii) $\nabla_q (b_1 \nu b_2 \nu) = b_1 \nu \nabla_q b_2 \nu + b_2 \nu \nabla_q b_1 \nu$.

(iii) $\nabla_q \frac{b_1 \nu}{b_2 \nu} = \frac{\nabla_q b_1 \nu - b_1 \nu \nabla_q b_2 \nu}{b_2 \nu (1 - q)}$, $b_2 \nu \neq 0$.

(iv) $\nabla_q (\log b \nu) = \frac{\ln q \nu (\log b \nu)}{(q - 1) b \nu}$.

Ismail et al. [33] were first who discussed various properties of function theory by virtue of $q$-theory.

In [34], Kanas and Raducanu introduced an operator $R_q^h : \mathbb{A} \to \mathbb{A}$ defined by

$$R_q^h \nu = \nu + \sum_{j=1}^{\infty} [\nabla_q \nu]_j \frac{j + \lambda - 1}{[\lambda]_q!} \frac{a_j \nu^j}{[j - 1]_q!}, \quad (\lambda > -1),$$

where $h : \mathbb{A}$ and $*$ denotes the Hadamard product(convolution).

Also, for $\lambda = m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then the operator (2.6) can be written as:

$$R_q^m \nu = \frac{v \nabla_q \left( v^{m-1} \nu \right)}{[m]_q!}.$$ 

We note that $R_q^0 \nu = h(\nu)$ and $R_q^1 \nu = v \nabla_q h(\nu)$.

We obtain the Ruscheweyh derivative operator [35], in particular, for $q \to 1^-$.

The $q$–extension of the Srivastava-Attiya operator discussed by Shah and Noor in [36]. They defined, for $b \in \mathbb{C} \setminus \mathbb{Z},$ $s \in \mathbb{C}$ when $|v| < 1$ and $\mathcal{R}(s) > 1$ when $|v| = 1$, the operator $J_q^s : \mathbb{A} \to \mathbb{A}$ by

$$J_q^s \nu = \Lambda_q (s, b; \nu) \ast h(\nu).$$
where

\[ \Lambda_{q}(s, b; \nu) = \nu + \sum_{j=2}^{\infty} \left[ \frac{[1 + b]_{q}}{[j + b]_{q}} \right]^{s} a_{j} \nu^{j}. \]  

The \( q \)-Srivastava-Attiya operator \( J_{q,b}^{s} \) generalizes some well-known operators such as \( q \)-Alexander, \( q \)-Libera, \( q \)-Bernardi and Srivastava-Attiya operator, we refer to [37, 38].

From (1.5) and (1.6), we use the convolution technique to define \( \Upsilon_{q,b}^{s,t} : \mathfrak{H} \to \mathfrak{H} \) as follows:

\[ \Upsilon_{q,b}^{s,t} b(\nu) = \left( \mathcal{R}_{q}^{s} * J_{q,b}^{t} \right) b(\nu) \]

\[ = \nu + \sum_{j=1}^{\infty} \left[ \frac{[j + \lambda - 1]_{q}}{[\lambda]_{q}[j - 1]_{q}} \right]^{s} \left[ \frac{[1 + b]_{q}}{[j + b]_{q}} \right] a_{j} \nu^{j}. \]  

It is clear that

\[ \Upsilon_{q,b}^{0,1} b(\nu) = \mathcal{R}_{q}^{1} b(\nu) \text{ and } \Upsilon_{q,b}^{1,0} b(\nu) = J_{q,b}^{1} b(\nu) \]

The following identities can be implied from (1.5) to (1.7).

\[ \nu \nabla_{q} \left( \Upsilon_{q,b}^{s,t+1,\lambda} b(\nu) \right) = \left( 1 + \frac{[b]_{q}}{q^{b}} \right) \Upsilon_{q,b}^{s,t} b(\nu) - \frac{[b]_{q}}{q^{b}} \Upsilon_{q,b}^{s+1,t} b(\nu). \]  

\[ \nu \nabla_{q} \left( \Upsilon_{q,b}^{s,t+1} b(\nu) \right) = \left( 1 + \frac{[\lambda]_{q}}{q^{b}} \right) \Upsilon_{q,b}^{s,t+1} b(\nu) - \frac{[\lambda]_{q}}{q^{b}} \Upsilon_{q,b}^{s,t+1} b(\nu). \]  

Several scholars studied various geometrical properties of analytic functions associated with \( q \)-linear operators, we refer to readers [39, 40, 41]. Now, we use the \( q \)-difference operator and the fuzzy subordination principle to define certain new subclasses \( \mathcal{F} \mathcal{S}_{q} (\varrho, \delta; g) \) and \( \mathcal{F} \mathcal{C}_{q} (\varrho, \delta; g) \) as the following.

For \( \varrho \in \mathbb{R} : |\varrho| < \frac{\pi}{2}, q \in (0, 1), 0 \neq \delta \in \mathbb{C}, \nu \in \Pi \) and \( g \in \mathfrak{H} \), where \( \mathfrak{H} \) be the class of all functions \( g \) which are analytic and univalent in \( \Pi \), and for which \( g(0) = 1 \) and \( \text{Re} (g(\nu)) > 0 \) in \( \Pi \).

\[ \mathcal{F} \mathcal{S}_{q} (\varrho, \delta; g) = \left\{ b \in \mathfrak{H} : 1 + \frac{e^{i\varrho}}{\delta \cos \varrho} \left( \nu \nabla_{q} b - 1 \right) \prec g(\nu) \right\}, \]

\[ \mathcal{F} \mathcal{C}_{q} (\varrho, \delta; g) = \left\{ b \in \mathfrak{H} : 1 + \frac{e^{i\varrho}}{\delta \cos \varrho} \left( \frac{\nu \nabla_{q} b}{\nabla_{q} b} - 1 \right) \prec g(\nu) \right\}. \]

In application of the operator given in (1.7), we define;

\[ \mathcal{F} \mathcal{S} T_{q,b}^{s,t} (\varrho, \delta; g) = \left\{ b \in \mathfrak{H} : \Upsilon_{q,b}^{s,t} b(\nu) \in \mathcal{F} \mathcal{S}_{q} (\varrho, \delta; g) \right\}, \]

and

\[ \mathcal{F} \mathcal{C} T_{q,b}^{s,t} (\varrho, \delta; g) = \left\{ b \in \mathfrak{H} : \Upsilon_{q,b}^{s,t} b(\nu) \in \mathcal{F} \mathcal{C}_{q} (\varrho, \delta; g) \right\}, \]

where \( g \in \mathfrak{H}, \lambda > -1, q \in (0, 1), s, \varrho, \delta \in \mathbb{R} : |\varrho| < \frac{\pi}{2}, b > -1, 0 \neq \delta \in \mathbb{C} \) and \( \nu \in \Pi \).
It is obvious that

\[ h \in FCV_{q,b}^{\nu,\lambda}(\varrho,\delta;g) \text{ if and only if } \nu \nabla_q h \in FST_{q,b}^{\nu,\lambda}(\varrho,\delta;g). \]  

(1.10)

Particularly, when \( \delta = 1 \) and \( \varrho = 0 \), we obtain the classes \( FST_{q,b}^{\nu,\lambda}(g) \) and \( FCV_{q,b}^{\nu,\lambda}(g) \) introduced by the authors in [26]. For \( \lambda = 0 = s \), the classes \( FST_{q,b}^{\nu,\lambda}(g,\delta;g) \) and \( FCV_{q,b}^{\nu,\lambda}(g,\delta;g) \) reduces to the classes \( FS_q(g,\delta;g) \) and \( FC_q(g,\delta;g) \) respectively. Moreover, when \( q \to 1^- \), we have the classes, \( FST(g,\delta;g) \) and \( FCV(g,\delta;g) \), introduced by the authors in [27]. Furthermore, for \( \delta = 1 \) and \( \varrho = 0 \), the classes \( FS(g,\delta;g) \) and \( FC(g,\delta;g) \) coincides with the classes \( FST(g) \) and \( FC(g) \), introduced and studied by Shah et al. [20].

Now, in the next section, the inclusion problems between the subclasses are investigated. Furthermore, we show that the subclasses preserve under \( q \)-Bernardi integral operator. We need the following lemma for our findings.

**Lemma 1.1.** [25] Let \( \beta \) and \( \gamma \) be complex numbers with \( \beta \neq 0 \) and let \( g(\nu) \) be a convex univalent in \( \Pi \) with \( g(0) = 1 \) and

\[ Re\{\beta g(\nu) + \gamma\} > 0. \]  

(1.11)

If \( p(\nu) = 1 + p_1 \nu + p_2 \nu^2 + \ldots \) is analytic in \( \Pi \), then

\[ p(\nu) + \frac{\nu \nabla_q p(\nu)}{\beta p(\nu) + \gamma} \preceq_{\varphi} g(\nu) \text{ implies } p(\nu) \preceq_{\varphi} g(\nu), \]

where \( \varphi : \mathbb{C} \to [0,1] \).

## 2. Main Results

### 2.1. Inclusion Results

**Theorem 2.1.** Let \( g \in \mathcal{H}, \varrho \in (0,1), \lambda \in \mathbb{N}_0, s, \varrho \in \mathbb{R} : |\varrho| < \frac{\pi}{2}, b > -1 \) and \( 0 \neq \delta \in \mathbb{C} \). Then,

\[ FST_{q,b}^{\nu,\lambda}(\varrho,\delta;g) \subset FST_{q,b}^{\nu+1,\lambda}(\varrho,\delta;g), \]

(2.1)

for

\[ \Re \{e^{i\nu} \delta \cos \varrho (g(\nu) - 1) + (1 + x_q)\} > 0, \text{ with } x_q = \frac{[b]_q}{q^b}, \]  

(2.2)

and

\[ FST_{q,b}^{\nu+1,\lambda+1}(\varrho,\delta;g) \subset FST_{q,b}^{\nu,\lambda}(\varrho,\delta;g), \]

(2.3)

for

\[ \Re \{e^{i\nu} \delta \cos \varrho (g(\nu) - 1) + (1 + d_q)\} > 0, \text{ with } d_q = \frac{[\lambda]_q}{q^b}. \]  

(2.4)

**Proof.** To prove the relation (2.3), we suppose \( h \in FST_{q,b}^{\nu,\lambda}(\varrho,\delta;g) \). For analytic \( p_1(\nu) \) in \( \Pi \) with \( p_1(0) = 1 \), we set

\[ p_1(\nu) = \frac{1}{\delta \cos \varrho} \left\{ e^{\nu \nabla_q} \left( \frac{\gamma_{q,b}^{\nu+\lambda+1} h(\nu)}{\gamma_{q,b}^{\nu+\lambda} h(\nu)} - (1 - \delta) \cos \varrho - i \sin \varrho \right) \right\}. \]  

(2.5)
The identity (1.8) and (2.5) imply that

\[ p_1(\varrho) = \frac{1}{\delta \cos \varrho} \left[ e^{i\delta} \left( 1 + \frac{[b]_q}{q^b} \right) \frac{T_{q,b}^{s+1,\lambda}(\varrho)}{T_{q,b}^{s,\lambda}(\varrho)} - \frac{[b]_q}{q^b} \right] - (1 - \delta) \cos \varrho - i \sin \varrho, \]

equivalently

\[ \left( 1 + x_q \right) \frac{T_{q,b}^{s,\lambda}(\varrho)}{T_{q,b}^{s+1,\lambda}(\varrho)} = e^{-i\delta} \cos \varrho (p_1(\varrho) - 1) + \left( 1 + x_q \right), \quad \text{for } x_q = \frac{[b]_q}{q^b}. \]

The \( q \)-logarithmic differentiation and (2.5) yield

\[ \frac{1}{\delta \cos \varrho} \left[ e^{i\delta} v \nabla_q \left( \frac{T_{q,b}^{s,\lambda}(\varrho)}{T_{q,b}^{s,\lambda}(\varrho)} \right) - (1 - \delta) \cos \varrho - i \sin \varrho \right] = p_1(\varrho) + \frac{v \nabla_q p_1(\varrho)}{e^{-i\delta} \cos \varrho (p_1(\varrho) - 1) + \left( 1 + x_q \right)}. \quad (2.6) \]

Since \( h \in \mathbb{F} S T_{q,b}^{s,\lambda}(\varrho, \delta; g) \), from (2.6) we have

\[ p_1(\varrho) + \frac{v \nabla_q p_1(\varrho)}{e^{-i\delta} \cos \varrho (p_1(\varrho) - 1) + \left( 1 + x_q \right)} \lessgtr g(\varrho), \quad (2.7) \]

for \( g \in \mathbb{M} \), we assume that

\[ \Re \left\{ e^{-i\delta} \cos \varrho (p_1(\varrho) - 1) + \left( 1 + x_q \right) \right\} > 0, \]

by Lemma 1.1 and (2.7), we conclude \( p_1(\varrho) \lessgtr g(\varrho) \) implies \( h \in \mathbb{F} S T_{q,b}^{s+1,\lambda}(\varrho, \delta; g) \).

To prove (2.5), we set, for analytic \( p_2(\varrho) \) in \( \Pi \) with \( p_2(0) = 1 \),

\[ p_2(\varrho) = \frac{1}{\delta \cos \varrho} \left[ e^{i\delta} v \nabla_q \left( \frac{T_{q,b}^{s,\lambda}(\varrho)}{T_{q,b}^{s+1,\lambda}(\varrho)} \right) - (1 - \delta) \cos \varrho - i \sin \varrho \right]. \quad (2.8) \]

Now, on the similar techniques as used before, we can easily obtain the required result by using the identity (1.9) along with Lemma 1.1.

particularly, if we take \( \delta = 1 \) and \( \varrho = 0 \), we have

**Corollary 2.1.** [26] Let \( q \in (0, 1) \), \( \lambda \in \mathbb{N}_0 \), \( s \in \mathbb{R} \), \( b > -1 \), and \( g \in \mathbb{M} \). Then,

\[ \mathbb{F} S T_{q,b}^{s,\lambda}(g) \subset \mathbb{F} S T_{q,b}^{s+1,\lambda}(g), \]

for

\[ \Re \left\{ g(\varrho) + x_q \right\} > 0, \quad \text{with } x_q = \frac{[b]_q}{q^b}, \]

and

\[ \Re \left\{ g(\varrho) + d_q \right\} > 0, \quad \text{with } d_q = \frac{[\lambda]_q}{q^b}. \]
Furthermore, if we choose \( \lambda = 0 \) and \( q \to 1^- \) then the inclusion relation (2.3) is reduced to the following result.

**Corollary 2.2.** [20] Let \( s \in \mathbb{R}, b > -1, \) and \( g \in \mathfrak{M} \). Then, for \( \Re \{g(v) + b\} > 0, \)
\[
\mathbb{F} \mathbb{S} T_s^b \mathbb{F}_g \subset \mathbb{F} \mathbb{S} T_{b+1}^s \mathbb{F}_g.
\]

**Theorem 2.2.** Let \( q \in (0, 1), \lambda \in \mathbb{N}_0, s, q \in \mathbb{R} : |q| < \frac{\pi}{2}, b > -1, 0 \neq \delta \in \mathbb{C} \) and \( g \in \mathfrak{M} \). Then, for the conditions (2.2) and (2.4),
\[
\mathbb{F} \mathbb{C} \mathbb{V}^{s, \lambda}_{q,b} (q, \delta; g) \subset \mathbb{F} \mathbb{C} \mathbb{V}^{s+1, \lambda}_{q,b} (q, \delta; g),
\]  
and
\[
\mathbb{F} \mathbb{C} \mathbb{V}^{s, \lambda+1}_{q,b} (q, \delta; g) \subset \mathbb{F} \mathbb{C} \mathbb{V}^{s, \lambda}_{q,b} (q, \delta; g),
\]

respectively.

**Proof.** Let \( h \in \mathbb{F} \mathbb{C} \mathbb{V}^{s, \lambda}_{q,b} (q, \delta; g) \). Then, by (1.10), \( \nu \nabla_q h \in \mathbb{F} \mathbb{S} T_{q,b}^s (q, \delta; g) \). This implies, by using Theorem 2.1, \( \nu \nabla_q h \in \mathbb{F} \mathbb{S} T_{q,b}^{s+1} (q, \delta; g) \). Again, by (1.10), we get \( h \in \mathbb{F} \mathbb{C} \mathbb{V}^{s, \lambda+1}_{q,b} (q, \delta; g) \). One can easily prove the relation (2.10) by using the same method as used for the relation (2.9). \( \Box \)

Particularly, if we take \( \delta = 1 \) and \( q = 0 \), we have

**Corollary 2.3.** [26] Let \( q \in (0, 1), \lambda \in \mathbb{N}_0, s \in \mathbb{R}, b > -1, \) and \( g \in \mathfrak{M} \). Then,
\[
\mathbb{F} \mathbb{C} \mathbb{V}^{s, \lambda}_{q,b} (g) \subset \mathbb{F} \mathbb{C} \mathbb{V}^{s+1, \lambda}_{q,b} (g),
\]
for
\[
\Re \{g(v) + x_q\} > 0, \text{ with } x_q = \frac{[b]_q}{q},
\]
and
\[
\mathbb{F} \mathbb{C} \mathbb{V}^{s, \lambda+1}_{q,b} (g) \subset \mathbb{F} \mathbb{C} \mathbb{V}^{s, \lambda}_{q,b} (g),
\]
for
\[
\Re \{g(v) + d_q\} > 0, \text{ with } d_q = \frac{[\lambda]_q}{q}.
\]

Furthermore, if we choose \( \lambda = 0 \) and \( q \to 1^- \) then the inclusion relation (2.9) is reduced to the following result.

**Corollary 2.4.** [20] Let \( s \in \mathbb{R}, b > -1, \) and \( g \in \mathfrak{M} \). Then, for \( \Re \{g(v) + b\} > 0, \)
\[
\mathbb{F} \mathbb{C} \mathbb{V}^s_b (g) \subset \mathbb{F} \mathbb{C} \mathbb{V}^{s+1}_b (g).
\]

### 2.2. Integral Preserving Property

**Theorem 2.3.** Let \( g \in \mathfrak{M}, q \in (0, 1), \lambda \in \mathbb{N}_0, s, q \in \mathbb{R} : |q| < \frac{\pi}{2}, b > -1, 0 \neq \delta \in \mathbb{C} \), and \( \mathbb{F}_{q,b} \) is defined by
\[
\mathbb{F}_{q,b}(v) = \frac{[1 + b]_q}{q^b} \int_0^v t^{b-1} \mathbb{F}_q(t) \nabla_q t.
\]

Then, for
\[
\Re \left\{ e^{-i\delta} \cos q (g(v) - 1) + (1 + [b]_q) \right\} > 0,
\]
\[
\mathbb{F}_{q,b} \in \mathbb{F} \mathbb{S} T_{q,b}^{s, \lambda} (q, \delta; g) \] whenever \( h \in \mathbb{F} \mathbb{S} T_{q,b}^{s, \lambda} (q, \delta; g) \).
\textbf{Proof.} Let $h \in \mathbb{F}ST_{q,b}^{s,\lambda}(\varphi)$ and consider
\begin{equation}
\chi(\nu) = \frac{1}{\delta \cos \varrho} \left\{ e^{i\varrho} \nu \nabla_q \left( \frac{\Gamma_{q,b}^{s,\lambda} \varphi(v)}{\Gamma_{q,b}^{s,\lambda} \varphi(v)} \right) - (1 - \delta) \cos \varrho - i \sin \varrho \right\}, \tag{2.12}
\end{equation}
with $\chi(\nu)$ is analytic in $\Pi$ with $\chi(0) = 1$.

From (2.11), we can write
\begin{equation}
\nabla_q \left( \nu b_{q,b}(\nu) \right) = \nu^{b-1} h(\nu).
\end{equation}
Using the product rule of $q$–difference operator, we get
\begin{equation}
\nu \nabla_q \varphi_{q,b}(\nu) = \left( 1 + \frac{[b]_q}{q^b} \right) h(\nu) - [b]_q \varphi_{q,b}(\nu). \tag{2.13}
\end{equation}
From (2.9), (2.10) and (1.5), we have
\begin{equation}
\left( 1 + \frac{[b]_q}{q^b} \right) \frac{J_{q,b}^{s,\lambda} \varphi(\nu)}{J_{q,b}^{s,\lambda} \varphi(\nu)} = e^{-i\varrho} \delta \cos \varrho \left( \chi(\nu) - 1 \right) + \left( 1 + [b]_q \right).
\end{equation}
After $q$–logarithmic differentiation, we get
\begin{equation}
\frac{1}{\delta \cos \varrho} \left\{ e^{i\varrho} \nu \nabla_q \left( \frac{\Gamma_{q,b}^{s,\lambda} \varphi(\nu)}{\Gamma_{q,b}^{s,\lambda} \varphi(\nu)} \right) - (1 - \delta) \cos \varrho - i \sin \varrho \right\} = \chi(\nu) + \frac{\nu \nabla_q \chi(\nu)}{e^{-i\varrho} \delta \cos \varrho \left( \chi(\nu) - 1 \right) + \left( 1 + [b]_q \right)} \prec \varphi(\nu),
\end{equation}
we have used the fact $h \in \mathbb{F}ST_{q,b}^{s,\lambda}(\varphi, \delta; g)$. Since $g \in \mathbb{M}$ and we assume that $\Re \left\{ e^{-i\varrho} \delta \cos \varrho \left( \varphi(\nu) - 1 \right) + \left( 1 + [b]_q \right) \right\} > 0$, by using Lemma 1.1, we conclude that $\chi(\nu) \prec \varphi(\nu)$ and this completes the proof. \hfill \ensuremath{\square}

In particular, when $\delta = 1$ and $\varrho = 0$, we have

**Corollary 2.5.** [26] Let $h \in \mathbb{F}ST_{q,b}^{s,\lambda}(g)$. Then, $\varphi_{q,b}(\nu)$ is in $\mathbb{F}ST_{q,b}^{s,\lambda}(g)$, where $\varphi_{q,b}(\nu)$ is given by (2.11).

Moreover, if we take $\lambda = 0$ and $q \to 1^-$ then we get the following result.

**Corollary 2.6.** [20] Let $h \in \mathbb{F}ST_{b}^{s,\lambda}(g)$. Then, $\varphi_{q,b}(\nu)$ is in $\mathbb{F}ST_{b}^{s,\lambda}(g)$, where $\varphi_{q,b}(\nu)$ is given by (2.11).

**Remark 2.1.** (i) On following the same method as used in Theorem 2.3, we can easily prove that the integral operator, given by (2.11), preserves the class $\mathbb{F}CV_{q,b}^{s,\lambda}(\varphi, \delta; g)$.

(ii) Particularly, the classes $\mathbb{F}CV_{q,b}^{s,\lambda}(g)$ and $\mathbb{F}CV_{b}^{s,\lambda}(\varphi)$ defined in [26] and [20], respectively, are invariant under the $q$–Bernardi integral operator.
3. Conclusions

In this article, we have defined q-analogue of certain subclasses of univalent functions with the help of fuzzy subsets. The $q$–Ruscheweyh derivative operator and the $q$–Srivastava-Attiya operator are combined by the Hadamard product and then the resultant operator applied on to the newly defined classes to obtain some generalized subclasses. We presented various classical results, such as the inclusion relationships and integral preserving property, for our newly defined subclasses. Various previous work has pointed out as the corollaries of our main investigations.

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Conflict of Interest

The authors declare no conflict of interest.

References


