Blow-up in a $p$-Laplacian mutualistic model on graphs

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Abstract: In this paper, a $p$-Laplacian ($p > 2$) reaction-diffusion system on weighted graphs is introduced to describe a network mutualistic model of population ecology. After overcoming difficulties caused by the nonlinear $p$-Laplacian, we propose a new strong mutualistic condition, and it is proved that the solution of this system blows up for any nontrivial initial data under this condition. In this sense, we extend the blow-up results of models with graph Laplacian ($p = 2$) in [6] to general graph $p$-Laplacian.

Keywords: $p$-Laplacian; network; blow-up; strong mutualistic; comparison principle.

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1 Introduction

In recent years, reaction-diffusion systems on complex networks have been studied extensively, for example, in epidemic processes or population ecology [6, 7, 8, 9, 12]. A network is mathematically described as a undirected graph $G = (\Omega, E)$, which contains a set $\Omega$ of vertices and a set $E$ of edges $(x, y)$ connecting vertex $x$ and vertex $y$. If vertices $x$ and $y$ are connected by an edge (also called adjacent), we write $x \sim y$. $G$ is called a finite-dimensional graph if it has a finite number of edges and vertices. A graph is weighted if each adjacent $x$ and $y$ is assigned a weight function $\omega(x, y)$. Here $\omega : \Omega \times \Omega \to [0, +\infty)$ satisfies that $\omega(x, y) = \omega(y, x)$ and $\omega(x, y) > 0$ if and only if $x \sim y$. Throughout this paper, $G = (\Omega, E)$ is assumed to be a weighted finite-dimensional graph with $\Omega = \{1, 2, \ldots, n\}$.

In order to describe our problem more conveniently, we first introduce the following discrete $p$-Laplacian operators defined on a network.

Definition 1.1 For a function $u : \Omega \to \mathbb{R}$ and $p \in (2, +\infty)$, the discrete $p$-Laplacian $\Delta^p_\omega$ on $\Omega$ is defined by

$$\Delta^p_\omega u(x) := \sum_{y \sim x, y \in \Omega} |u(y) - u(x)|^{p-2}(u(y) - u(x))\omega(x, y).$$  \hspace{1cm} (1.1)

When $p = 2$, it is called the discrete Laplacian $\Delta_\omega := \Delta^2_\omega$ on $\Omega$, which is defined by

$$\Delta_\omega u(x) := \sum_{y \sim x, y \in \Omega} (u(y) - u(x))\omega(x, y).$$  \hspace{1cm} (1.2)

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In particular, in the case of condition (1.4) holds, it is proved that the solution of (1.3) blows up for any nontrivial initial data \( p > 2 \). When condition (1.4) holds, it is proved that the solution of (1.3) blows up for any nontrivial initial data \( p > 2 \) is actually nonlinear, which is different from the classical Laplacian \( \Delta \) or the discrete Laplacian \( \Delta_\omega \).

We are mainly interested in studying the blow-up properties for the solution of the following mutualistic model with \( p \)-Laplacian \( (p > 2) \) defined on networks

\[
\begin{align*}
\frac{\partial u_1}{\partial t} - d_1 \Delta_p u_1 &= u_1(a_1 - b_1 u_1 + c_1 u_2), \quad (x, t) \in \Omega \times (0, +\infty), \\
\frac{\partial u_2}{\partial t} - d_2 \Delta_p u_2 &= u_2(a_2 + c_2 u_1 - b_2 u_2), \quad (x, t) \in \Omega \times (0, +\infty), \\
u_1(x, 0) &\triangleq u_{10}(x) \geq (\neq)0, \quad u_2(x, 0) \triangleq u_{20}(x) \geq (\neq)0, \quad x \in \Omega.
\end{align*}
\]

Here \( u_i \) represent the spatial density of the \( i \)th species at time \( t \) and \( d_i \) represents its respective diffusion rate. The nonnegative constant \( a_i \) is the birth rate, \( b_i \) is its respective intraspecific competition, and the parameters \( c_i \) is interspecific cooperations of the \( i \)th species.

If \( \Delta_p \) is replaced by classical Laplacian in (1.3), the strong mutualistic \( (b_1/c_1 < c_2/b_2) \) population dynamical system occurs blow-up if the intrinsic growth rates of population are large or the initial data is sufficiently large [4]. When \( p = 2 \) in (1.3), Liu, Chen and Tian [6] proved that the solution blows up for any nontrivial initial data satisfying that for all \( x \in \Omega \), \( \min\{u_{10}(x), u_{20}(x)\} \neq 0 \), under the strong mutualistic condition \( b_1/c_1 < c_2/b_2 \) and \( \min\{a_1/d_1, a_2/d_2\} \geq 1 \).

In this paper, when \( p > 2 \), we overcome the difficulties caused by the nonlinear operators \( p \)-Laplacian \( \Delta_p \) and study the blow-up properties for the solution of system (1.3). First, we prove the Green formula of nonlinear operators \( \Delta_p \) and consider the eigenvalue problem \( \Delta_p \). Second, with the help of the following inequality (see Lemma 2.4)

\[
|b - a|^{p-2}(b - a) \leq 2^{p-2}[|b|^{p-2}b - |a|^{p-2}a] \quad \text{with} \quad b \geq a.
\]

the comparison principle of system (1.3) is constructed (see Theorem 2.5). Finally, we propose a new strong mutualistic condition

\[
\frac{b_1}{c_1} < \left(\frac{d_1}{d_2}\right)^{\frac{1}{p-2}} < \frac{c_2}{b_2}. \tag{1.4}
\]

When condition (1.4) holds, it is proved that the solution of (1.3) blows up for any nontrivial initial data satisfying that for all \( x \in \Omega \), \( \min\{u_{10}(x), u_{20}(x)\} \neq 0 \) (see Theorem 3.2).

## 2 Preliminaries

**Lemma 2.1 (Green Formula).** For any functions \( u, v : \Omega \to \mathbb{R} \), the \( p \)-Laplacian \( \Delta_p \) satisfies that

\[
2 \sum_{x \in \Omega} v(x)(-\Delta_p u)(x) = \sum_{x,y \in \Omega} |u(y) - u(x)|^{p-2}(u(y) - u(x))(v(y) - v(x))\omega(x, y). \tag{2.1}
\]

In particular, in the case of \( u = v \), we have

\[
2 \sum_{x \in \Omega} u(x)(-\Delta_p u)(x) = \sum_{x,y \in \Omega} |u(y) - u(x)|^p\omega(x, y). \tag{2.2}
\]
Proof Using (1.1), we have
\[
\sum_{x \in \Omega} v(x)(-\Delta_p^x)u(x) = - \sum_{x \in \Omega} v(x) \sum_{y \sim x, y \in \Omega} |u(y) - u(x)|^{p-2}(u(y) - u(x))\omega(x, y)
\]
\[
= - \sum_{x, y \in \Omega} v(x)|u(y) - u(x)|^{p-2}(u(y) - u(x))\omega(x, y).
\]  
(2.3)

Meanwhile, we also deduce that
\[
\sum_{x \in \Omega} v(x)(-\Delta_p^x)u(x) = - \sum_{x, y \in \Omega} v(y)|u(y) - u(x)|^{p-2}(u(x) - u(y))\omega(x, y)
\]
\[
= \sum_{x, y \in \Omega} v(y)|u(y) - u(x)|^{p-2}(u(x) - u(y))\omega(x, y).
\]  
(2.4)

Consequently, combining (2.3) and (2.4) yields
\[
2 \sum_{x \in \Omega} v(x)(-\Delta_p^x)u(x) = \sum_{x, y \in \Omega} |u(y) - u(x)|^{p-2}(u(y) - u(x))(v(y) - v(x))\omega(x, y),
\]
which completes the proof. \(\square\)

Lemma 2.2 Consider the eigenvalue problem
\[
\begin{aligned}
-\Delta_p^x \phi(x) &= \lambda \phi(x), \quad x \in \Omega, \\
\sum_{x \in \Omega} \phi(x) &= 1.
\end{aligned}
\]  
(2.5)

There exists
\[
\lambda_1 := \min_{\phi \neq 0} \frac{\sum_{x, y \in \Omega} |\phi(y) - \phi(x)|^p\omega(x, y)}{2 \sum_{x \in \Omega} \theta^2} \quad \text{for} \quad \phi : \Omega \to \mathbb{R}
\]  
(2.6)

and \(\Phi_1(x) > 0\) in \(\Omega\) satisfying the above system (2.5), which are called the first eigenvalue and eigenfunction of (2.5), respectively. Moreover, \(\lambda_1 = 0\).

Proof Multiplying the first equation of (2.5) by \(\phi\) and integrating over \(\Omega\), we have
\[
\sum_{x \in \Omega} \phi(x)(-\Delta_p^x)\phi(x) = \sum_{x \in \Omega} \lambda \phi^2.
\]

By (2.2), we deduce that
\[
\lambda = \frac{\sum_{x, y \in \Omega} |\phi(y) - \phi(x)|^p\omega(x, y)}{2 \sum_{x \in \Omega} \theta^2}.
\]

Hence we obtain
\[
\lambda_1 := \min_{\phi \neq 0} \frac{\sum_{x, y \in \Omega} |\phi(y) - \phi(x)|^p\omega(x, y)}{2 \sum_{x \in \Omega} \theta^2},
\]
where the minimum can be attained by taking \(\Phi_1 = \frac{1}{n}\), where \(n\) is the number of vortices in \(\Omega\), and \(\Phi_1\) satisfies \(\sum_{x \in \Omega} \Phi_1(x) = 1\). Therefore, by taking \(\Phi_1 = \frac{1}{n}\), we can get \(\lambda_1 = 0\), the proof is completed. \(\square\)

Definition 2.3 For any \(T > 0\), suppose that for each \(x \in \Omega\), \(\tilde{u}_1(x, \cdot), \tilde{u}_2(x, \cdot) \in C([0, T])\) are differentiable in \([0, T]\). If \((\tilde{u}_1, \tilde{u}_2)\) satisfies
\[
\begin{cases}
\frac{\partial \tilde{u}_1}{\partial t} - d_1 \Delta_p^x \tilde{u}_1 \leq \tilde{u}_1(a_1 - b_1 \tilde{u}_1 + c_1 \tilde{u}_2), & (x, t) \in \Omega \times (0, T], \\
\frac{\partial \tilde{u}_2}{\partial t} - d_2 \Delta_p^x \tilde{u}_2 \leq \tilde{u}_2(a_2 + c_2 \tilde{u}_1 - b_2 \tilde{u}_2), & (x, t) \in \Omega \times (0, T], \\
\tilde{u}_1(x, 0) \leq u_{10}(x), & \tilde{u}_2(x, 0) \leq u_{20}(x), \quad x \in \Omega,
\end{cases}
\]  
(2.7)

\((\tilde{u}_1, \tilde{u}_2)\) is called a lower solution of (1.3) on \(\Omega \times [0, T]\). Moreover, if \((\tilde{u}_1, \tilde{u}_2)\) satisfies (2.7) by reversing all the inequalities, \((\tilde{u}_1, \tilde{u}_2)\) is called an upper solution of (1.3) on \(\Omega \times [0, T]\).
Lemma 2.4 (Lemma B.4 in [2]) For $p > 2$, $J_p(t) := |t|^{p-2}t$, we have
\[ 2^{2-p}|b-a|^p \leq (J_p(b) - J_p(a))(b-a), \quad a, b \in \mathbb{R}. \]
Moreover, if $b \geq a$, we have
\[ J_p(b-a) \leq 2^{p-2}[J_p(b) - J_p(a)]. \quad (2.8) \]

With the help of inequality (2.8), we propose the following comparison principle.

Theorem 2.5 (Comparison Principle). Assume that $(u_1, u_2)$ is a solution to the system (1.3). If $(\hat{u}_1, \hat{u}_2)$ is a lower solution of (1.3) on $\Omega \times [0, T]$, then $(u_1, u_2) \geq (\hat{u}_1, \hat{u}_2)$ in $\Omega \times [0, T]$.

Proof Denote $z_1 := (u_1 - \hat{u}_1)e^{-Kt}$ and $z_2 := (u_2 - \hat{u}_2)e^{-Kt}$, where $K$ is a positive constant to be determined later. Notice that $z_i(x, t) (i = 1, 2)$ are continuous on $[0, T]$ for each $x \in \Omega$ and $\Omega$ is finite, we can find $(x_0, t_0) \in \Omega \times [0, T]$ such that
\[ z_1(x_0, t_0) = \min_{x \in \Omega} \min_{t \in [0, T]} z_1(x, t), \quad (2.9) \]
which immediately implies that
\[ z_1(x_0, t_0) \leq z_1(y, t_0), \quad \text{for any } y \in \Omega. \]
This is equivalent to
\[ u_1(x_0, t_0) - \hat{u}_1(x_0, t_0) \leq u_1(y, t_0) - \hat{u}_1(y, t_0), \quad \text{for any } y \in \Omega. \quad (2.10) \]
and
\[ u_1(y, t_0) - u_1(x_0, t_0) \geq \hat{u}_1(y, t_0) - \hat{u}_1(x_0, t_0), \quad \text{for any } y \in \Omega. \quad (2.11) \]
Recalling the definition of $\Delta^p_p$, we have
\[ \Delta^p_p z_1(x_0, t_0) \geq 0. \quad (2.12) \]
At the same time, from the differentiability of $z_1(x, t)$ in $(0, T]$, we obtain
\[ \frac{\partial z_1}{\partial t}(x_0, t_0) \leq 0. \quad (2.13) \]
Note that
\[ \Delta^p_p z_1(x, t) = e^{-Kt(p-1)}\Delta^p_p(u_1 - \hat{u}_1)(x, t) \\
= e^{-Kt(p-1)} \sum_{y \sim x, y \in \Omega} \left| (u_1(y, t) - \hat{u}_1(y, t)) - (u_1(x, t) - \hat{u}_1(x, t)) \right|^{p-2} \\
\times \left[ (u_1(y, t) - \hat{u}_1(y, t)) - (u_1(x, t) - \hat{u}_1(x, t)) \right] \omega(x, y) \quad (2.14) \]
\[ = e^{-Kt(p-1)} \sum_{y \sim x, y \in \Omega} \left| (u_1(y, t) - u_1(x, t)) - (\hat{u}_1(y, t) - \hat{u}_1(x, t)) \right|^{p-2} \\
\times \left[ (u_1(y, t) - u_1(x, t)) - (\hat{u}_1(y, t) - \hat{u}_1(x, t)) \right] \omega(x, y), \]
we have
\[
\Delta^p u_1 - \tilde{u}_1(x_0, t_0) = \sum_{y \sim x_0, y \in \Omega} \left[ (u_1(y, t_0) - u_1(x_0, t_0)) - (\tilde{u}_1(y, t_0) - \tilde{u}_1(x_0, t_0)) \right]^{p-2} \left[ (u_1(y, t_0) - u_1(x_0, t_0)) - (\tilde{u}_1(y, t_0) - \tilde{u}_1(x_0, t_0)) \right] \omega(x_0, y).
\] (2.15)

Denote \( b_y := u_1(y, t_0) - u_1(x_0, t_0), \ a_y := \tilde{u}_1(y, t_0) - \tilde{u}_1(x_0, t_0) \) and \( J_p(t) := |t|^{p-2} t \).

In view of (2.11), we have \( b_y \geq a_y \) for any \( y \sim x_0 \) and \( y \in \Omega \). Combining this with (2.8) in Lemma 2.4, we deduce that
\[
|b_y - a_y|^{p-2} (b_y - a_y) = J_p(b_y - a_y) \leq 2^{p-2}|J_p(b_y) - J_p(a_y)| = 2^{p-2}|b_y|^{p-2}b_y - |a_y|^{p-2}a_y,
\]

which implies
\[
\Delta^p u_1 - \tilde{u}_1(x_0, t_0) = \sum_{y \sim x_0, y \in \Omega} |b_y - a_y|^{p-2} (b_y - a_y) \omega(x_0, y)
\]
\[
\leq 2^{p-2} \sum_{y \sim x_0, y \in \Omega} \left[ |b_y|^{p-2} b_y - |a_y|^{p-2} a_y \right] \omega(x_0, y)
\]
\[
= 2^{p-2} \left[ \sum_{y \sim x_0, y \in \Omega} |b_y|^{p-2} b_y \omega(x_0, y) - \sum_{y \sim x_0, y \in \Omega} |a_y|^{p-2} a_y \omega(x_0, y) \right]
\] (2.16)

Combining (2.16) with (2.14), we have
\[
\Delta^p z_1(x_0, t_0) \leq 2^{p-2} e^{-Kt_0(p-1)} \left[ \Delta^p u_1(x_0, t_0) - \Delta^p \tilde{u}_1(x_0, t_0) \right]
\] (2.17)

Note that \( (u_1, u_2) \) and \( (\tilde{u}_1, \tilde{u}_2) \) are a solution and a lower solution to the system (1.3), respectively. That is \( (u_1, u_2) \) and \( (\tilde{u}_1, \tilde{u}_2) \) satisfy
\[
\frac{\partial u_1}{\partial t} - d_1 \Delta^p u_1 \geq u_1(a_1 - b_1 u_1 + c_1 u_2)
\] (2.18)

and
\[
\frac{\partial \tilde{u}_1}{\partial t} - d_1 \Delta^p \tilde{u}_1 \leq \tilde{u}_1(a_1 - b_1 \tilde{u}_1 + c_1 \tilde{u}_2).
\] (2.19)

Recall that \( z_1 := (u_1 - \tilde{u}_1) e^{-Kt} \), we have
\[
\frac{\partial z_1}{\partial t} = -Kz_1 + e^{-Kt} \left( \frac{\partial u_1}{\partial t} - \frac{\partial \tilde{u}_1}{\partial t} \right)
\] (2.20)

Combining (2.18), (2.19) with (2.20), we obtain
\[
\frac{\partial z_1}{\partial t} \geq -Kz_1 + e^{-Kt} \left[ d_1 \Delta^p u_1 + u_1(a_1 - b_1 u_1 + c_1 u_2) - d_1 \Delta^p \tilde{u}_1 + \tilde{u}_1(a_1 - b_1 \tilde{u}_1 - c_1 \tilde{u}_2) \right]
\]
\[
= d_1 e^{-Kt} \left[ \Delta^p u_1 - \Delta^p \tilde{u}_1 \right] + (-K + a_1 - b_1 (u_1 + \tilde{u}_1) + c_1 u_2) z_1 + c_1 \tilde{u}_1 z_2
\] (2.21)

Combining (2.17) with (2.21), we deduce that
\[
2^{p-2} e^{-Kn_0(p-2)} \frac{\partial z_1}{\partial t}(x_0, t_0) - d_1 \Delta^p z_1(x_0, t_0) \geq 2^{p-2} e^{-Kt_0(p-2)} \left[ (-K + b_1) z_1(x_0, t_0) + b_1 z_2(x_0, t_0) \right],
\] (2.22)
we obtain that
\[ b_{11} := a_1 - b_1 (u_1(x_0, t_0) + \hat{u}_1(x_0, t_0)) + c_1 u_2(x_0, t_0), \quad b_{12} := c_1 \hat{u}_1(x_0, t_0) \quad (2.23) \]

Substituting (2.12) and (2.13) into (2.22), we deduce that
\[ ((-K + b_{11})z_1 + b_{12}z_2)(x_0, t_0) \leq 0. \quad (2.24) \]

Next we will prove \( z_1(x_0, t_0) \geq 0 \) by contradiction. Suppose that \( z_1(x_0, t_0) = -\delta < 0 \) on the contrary. Choosing
\[ K := \frac{|z_2(x_0, t_0)|}{\delta} |b_{12}(x_0, t_0)| + |b_{11}(x_0, t_0)| + 1, \]
we obtain that \(( -K + b_{11})z_1 + b_{12}z_2)(x_0, t_0) > 0\), which contradicts (2.24). Thus, we have \( z_1(x_0, t_0) \geq 0 \). In view of (2.9), it follows that \( z_1(x, t) \geq 0 \) for \((x, t) \in \Omega \times [0, T] \). By a similar argument to \( z_2 \), we can also obtain that \( z_2(x, t) \geq 0 \) for \((x, t) \in \Omega \times [0, T] \). These conclusions imply that \( u_i \geq \hat{u}_i \) \((i = 1, 2) \) for \((x, t) \in \Omega \times [0, T] \).

\( \square \)

3 Blow-up result

**Theorem 3.1** Let \( w(x, t) \) be a solution of
\[
\begin{aligned}
\frac{\partial w}{\partial t} - d\Delta^p w &= w(a + bw), \quad (x, t) \in \Omega \times (0, +\infty), \\
w(x, 0) &\geq (\neq)0, \quad x \in \Omega,
\end{aligned}
\quad (3.1)
\]
where \( d, a \) and \( b \) are constants and satisfying \( d > 0 \) and \( b > 0 \). Then we have the following blow-up properties:

(i) If \( a \geq 0 \), the solution of (3.1) blows up for any nontrivial initial data.

(ii) If \( a < 0 \), the solution of (3.1) blows up for large enough data.

**Proof** Define \( F(t) := \sum_{x \in \Omega} \Phi_1(x)w(x, t) \), where \( \Phi_1(x) \) is defined in Lemma 2.2. Deriving \( F(t) \) with respect to \( t \) and using (3.1), we have
\[
dF(t) = \sum_{x \in \Omega} \Phi_1(x)[d\Delta^p w + w(a + bw)].
\]

Note that \( \Phi_1(x) \) is a constant, due to (2.1) in Lemma 2.1, we have
\[
\sum_{x \in \Omega} \Phi_1(x)\Delta^p w = -\frac{1}{2} \sum_{x, y \in \Omega} |w(y) - w(x)|^{p-2}(w(y, t) - w(x, t))(\Phi_1(y) - \Phi_1(x))w(x, y) = 0.
\]

Hence, combining with \( \sum_{x \in \Omega} \Phi_1(x) = 1 \) in Lemma 2.2, we have
\[
dF(t) = \sum_{x \in \Omega} \Phi_1(x)w(a + bw) = aF(t) + b \sum_{x \in \Omega} \Phi_1(x)w^2
\leq aF(t) + b \sum_{x \in \Omega} \Phi_1(x)w^2 \sum_{x \in \Omega} \Phi_1(x) \geq aF(t) + b(\sum_{x \in \Omega} \Phi_1(x)w)^2 \quad (3.2)
= aF(t) + bF^2(t),
\]
where we use the Hölder’s inequality.

(i) In the case \( a \geq 0 \), from (3.2), we immediately obtain the blow-up result for any nontrivial initial data.

(ii) In the case \( a < 0 \), we can choose sufficiently large initial data \( w(x, 0) \) such that

\[
F(0) = \sum_{x \in \Omega} \Phi_1(x)w(x, 0) > -\frac{a}{b}.
\]

It follows from (3.2) that \( F(t) \) blows up for sufficiently large initial data.

**Theorem 3.2** If \( \frac{b_1}{c_1} < \left( \frac{d_1}{d_2} \right)^{\frac{1}{\gamma-1}} < \frac{b_2}{c_2} \) (strong mutualistic) holds, we have the following blow-up properties: The solution of (1.3) blows up for any nontrivial initial data satisfying that for all \( x \in \Omega \), \( \min \{u_{10}(x), u_{20}(x)\} \neq 0 \).

**Proof** We define \((\hat{u}_1(x, t), \hat{u}_2(x, t)) := (\delta_1 w(x, t), \delta_2 w(x, t))\), where the constants \( \delta_1, \delta_2 \) and function \( w(x, t) \) will be determined later. In order to guarantee \((\hat{u}_1(x, t), \hat{u}_2(x, t))\) to be a lower solution of (1.3), we need to show that \((\hat{u}_1(x, t), \hat{u}_2(x, t)) \leq (u_{10}(x), u_{20}(x)) \) and

\[
\begin{cases}
\frac{\partial w}{\partial t} - d_1 \delta_1^{-2} \Delta_\omega w \leq w(a_1 - b_1 \delta_1 w + c_1 \delta_2 w), & (x, t) \in \Omega \times (0, \infty), \\
\frac{\partial w}{\partial t} - d_2 \delta_2^{-2} \Delta_\omega w \leq w(a_2 + c_2 \delta_1 w - b_2 \delta_2 w), & (x, t) \in \Omega \times (0, \infty).
\end{cases}
\]  

(3.3)

Since the parameters satisfy \( \frac{b_1}{c_1} < \left( \frac{d_1}{d_2} \right)^{\frac{1}{\gamma-1}} < \frac{b_2}{c_2} \), by calculation, we get

\[
c_1 \left( \frac{d_1}{d_2} \right)^{\frac{1}{\gamma-1}} - b_1 > 0 \quad \text{and} \quad c_2 - b_2 \left( \frac{d_1}{d_2} \right)^{\frac{1}{\gamma-1}} > 0.
\]  

(3.4)

Thus, for sufficiently small positive constant \( \varepsilon \), we choose \( \delta_1 := \varepsilon, \delta_2 := \left( \frac{d_1}{d_2} \right)^{\frac{1}{\gamma-1}} \varepsilon \) such that \( d_1 \delta_1^{-2} = d_2 \delta_2^{-2} \),

\[
-b_1 \delta_1 + c_1 \delta_2 = \varepsilon \left[ c_1 \left( \frac{d_1}{d_2} \right)^{\frac{1}{\gamma-1}} - b_1 \right] > 0 \quad \text{and} \quad c_2 \delta_1 - b_2 \delta_2 = \varepsilon \left[ c_2 - b_2 \left( \frac{d_1}{d_2} \right)^{\frac{1}{\gamma-1}} \right] > 0.
\]  

(3.5)

Denote

\[
d := d_1 \delta_1^{-2} = d_2 \delta_2^{-2}, \quad A := \min \{a_1, a_2\} \geq 0, \quad B := \min \{-b_1 \delta_1 + c_1 \delta_2, c_2 \delta_1 - b_2 \delta_2\}.
\]  

(3.6)

To prove (3.3), it suffices to show

\[
\frac{\partial w}{\partial t} - d \Delta_\omega w \leq w(A + Bw), \quad (x, t) \in \Omega \times (0, T],
\]  

(3.7)

Hence \((\hat{u}_1(x, t), \hat{u}_2(x, t))\) is a lower solution of (1.3) provided \((\hat{u}_1(x, 0), \hat{u}_2(x, 0)) \leq (u_{10}(x), u_{20}(x))\). We choose the initial data of \( w \) as \( w(x, 0) := \min \{u_{10}(x), u_{20}(x)\} \) and \( \varepsilon \) small enough such that \( \delta_1, \delta_2 < 1 \), which guarantees that \((\hat{u}_1(x, 0), \hat{u}_2(x, 0)) \leq (u_{10}(x), u_{20}(x))\) holds.

Let \( w \) be a solution of

\[
\begin{cases}
d \frac{\partial w}{\partial t} - \Delta_\omega w = w(a + bw), & (x, t) \in \Omega \times (0, \infty), \\
w(x, 0) \geq (\neq) 0, & x \in \Omega.
\end{cases}
\]  

(3.8)

By applying Theorem 3.1 to \( w \), we have \( w(x, t) \) blows up, which implies \((\hat{u}_1(x, t), \hat{u}_2(x, t))\) blows up. By Lemma 2.2, the solution of system (1.3) blows up.

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