# Exponential decay of a Balakrishnan-Taylor plate with strong damping

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**Abstract**

In this manuscript, we study a thin and narrow plate equation modeling the deck of a suspension bridge subject to a Balakrishnan-Taylor damping and a strong one. First, by using the Faedo Galerkin method, we prove the existence of both global weak and regular solutions. Second, we prove the exponential stability of the energy for regular solutions by combining the multiplier method and a well-known result of Komornik.

**Keywords:** Plate, Balakrishnan-Taylor damping, strong damping, exponential decay.

**MSC 2020:** 35B40, 35L75, 74K10.

# 1 Introduction

A rectangular thin and narrow plate modeling the deck of a suspension bridge is considered in the domain $\Omega = (0,\pi)\times(-d,d)$, where $d << \pi$. The nonlocal evolution equation that describes how the plate is deformed looks like:

$$\begin{cases} p_{tt}(x,y,t) + \Delta^2 p(x,y,t) - \left(\phi(p) + \delta\langle p_x, p_{xt}\rangle\right) p_{xx} + \alpha\Delta^2 p_t = 0 & \text{in } \Omega\times(0,+\infty) \\ p(0,y,t) = p_{xx}(0,y,t) = p(\pi,y,t) = p_{xx}(\pi,y,t) = 0 & (y,t)\in(-d,d)\times(0,+\infty) \\ p_{yy}(x,\pm d,t) + \mu p_{xx}(x,\pm d,t) = 0 & (x,t)\in(0,\pi)\times(0,+\infty) \\ p_{yyy}(x,\pm d,t) + (2-\mu)p_{xxy}(x,\pm d,t) = 0 & (x,t)\in(0,\pi)\times(0,+\infty) \\ p(x,y,0) = p_0(x,y),\ p_t(x,y,0) = p_1(x,y) & \text{in } \Omega \end{cases} \tag{1.1}$$

where $\delta,\ \alpha > 0$ and $\phi$ which introduces a nonlocal effect is given by

$$\phi(p) = -a + b\int_\Omega p_x^2\,dxdy.$$

The constant $\mu$ is the Poisson ratio which is in general in the range $(-1,\frac{1}{2})$ due to physical reasons (see [4] for more details). It has a value of about 0.3 for metals and between 0.1

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and 0.2 for concrete. Due to this, we'll suppose that $0 < \mu < \frac{1}{2}$.
The constant $b > 0$ is determined by the elasticity of the deck's material, $b\int_{\Omega} p_x^2 \, dxdy$ determines the plate's geometric nonlinearity as a result of its stretching, and $a > 0$ is the constant of prestressing: if the plate is compressed, we have $a > 0$ and if the plate is stretched, one has $a < 0$.
We mean by $\langle \cdot, \cdot \rangle$ the usual scalar product in $L^2(\Omega)$.
Let us recall some works existing in the literature related to our problem. For one dimensional problem, in [3], the author considered the following equation

$$\begin{aligned} &p_{tt} + \alpha p_{xxxx} - \left(\beta + k\int_0^l \left[\frac{\partial p(\xi,t)}{\partial \xi}\right]^2 d\xi\right) p_{xx} + \gamma p_{xxxxt} \\ &-\mu\langle p_x, p_{tx}\rangle p_{xx} + \delta p_t = 0, \quad in\ (0,l)\times(0,+\infty), \end{aligned} \tag{1.2}$$

where the constants $\alpha, k, \gamma, \mu$ are positive, and the constants $\beta$ and $\delta$ have no restrictions on their sign. Here, $l$ denotes the beam's length. The author established existence, uniqueness, and regularity theorems for the situations where the beam's ends are clamped or hinged. In higher dimension, consider the work of Emmrich and Thalhammer [9], who provided a general model for describing nonlinear extensible beams with weak, viscous, strong, and Balakrishnan-Taylor damping as reads

$$\begin{aligned} &p_{tt} + \alpha\Delta^2 p + \xi p + \kappa p_t - \lambda\Delta p_t + \mu\Delta^2 p_t \\ &-\Big[\beta + \gamma\int_{\Omega}|\nabla p|^2 dx + \delta\left|\int_{\Omega}\nabla p\cdot\nabla p_t dx\right|^{q-2}\int_{\Omega}\nabla p\cdot\nabla p_t dx\Big]\Delta p = h \end{aligned} \tag{1.3}$$

in $\Omega\times(0,T)$, where $\Omega$ is a bounded domain and $T > 0$, the constants $\alpha$ and $\gamma$ are positive, and $\lambda, \mu$ and $\delta$ are nonnegative, whereas $\beta, \kappa, \xi \in \mathbb{R}$ and $q \geq 2$.
The authors demonstrated the existence of a weak solution for (1.3), under hinged either clamped boundary conditions, using time discretization in both cases: when $\lambda, \mu > 0$ and $q \geq 2$ or else when $\lambda = \mu = 0$ and $q = 2$. When $\kappa = \lambda = 0$ , $\mu > 0$ and $q = 2$ in (1.2), Clark [7] established existence, uniqueness and asymptotic behavior of the solutions in $N$-dimensional bounded and unbounded domains. In [29], You proved that there are global solutions in the cases where $\kappa = \mu = 0$, $\lambda > 0$, $q > 2$ and $\Omega = (0,1)$. Also, he gave results on the existence of inertial manifolds and on the finite-dimensional stabilization. Latter on, Tavares et al. [14] studied the problem (1.3) when $\lambda = \mu = 0, \kappa \in \mathbb{R}$ and $q \geq 2$, and they established the existence of a unique mild (and strong) solution, and analyzed the long-time dynamics of solutions (in the mentioned case) when $\kappa > 0$ and $\beta$ is bounded from below by a negative expression and with the existence of nonlinear source. We also mention the recent work [16] where the authors proved the stability of

rectangular Kirchhoff plates using the stochastic boundary element methods.
Now, let us mention some works on suspension bridges. In [13, 23], the existence of nonlinear oscillations was proved. The deck of a suspension bridge has been modelled in a simple form by [10]. See also Gazzola's book [12] and recent results [15, 20, 26] for additional details. The bending and stretching energies of the model presented in [10] were examined by Al-Gwaiz et al. in [1]. We mention also the recent work of [8] in which the authors gave a new model for a suspension bridge.
Recently, many researchers have been interested in studying the stability of a plate modeling the deck of a suspension bridge. Messaoudi et al. [24] demonstrated an exponential decay in the presence of both a global frictional damping and a nonlinear term. In [5] (resp. in [6]), the authors studied the same problem as in [24] but with linear (resp. nonlinear) local damping distributed around a neighborhood of the boundary, and they proved an exponential decay estimate of the associated energy.

Liu et al. [22] expanded the work of [27] and proved, without considering the relation between $m$ and $r$, that the solutions of the equation:

$$p_{tt} + \Delta^2 p + ap + |p_t|^{m-2} p_t = |p|^{r-2} p, \quad m \geq 2,\ r > 2,$$

exist globally if and only if there exists a real number $t_0 \in [0, T_{max})$ such that $p(t_0) \in W$ and the energy at the time $t_0$ is less than a such constant that depends on $r$ and $C_r$ ( $C_r$ is defined in (2.6)), where

$$T_{max} = \sup\{T > 0 : p = p(t) \text{ exists on } [0, T]\}$$

and

$$W = \{p \in N_+ : J(p) < d\},$$

with

$$N_+ = \{p \in V : I(p) > 0\} \cup \{0\}, I(p) = \|p\|_V^2 + (ap, p) - \|p\|_r^r,$$
$$J(p) = \frac{1}{2}\|p\|_V^2 + \frac{1}{2}(ap, p) - \frac{1}{r}\|p\|_r^r \text{ and } d = \inf_{p \in V \setminus \{0\}} \max_{\lambda > 0} J(\lambda p).$$

Moreover, an energy decay results were obtained, and when $r > m$ a blow-up result was established. Later on, in [25], the authors established the existence of a global weak solution and demonstrated a stability result, when an external force $f$ and a nonlinear frictional damping are presents. Finally, we cite the work [17] in which the author studied the same problem as here, but subject to a different types of damping: one of memory type (of the form $\int_0^t g(s)\Delta^2 p(s)ds$ and the other one is a nonlinear localized frictional damping (of the form $a(x,y)|p_t|^m p_t$). The author proved the existence of global solutions as well as a general stability result.

Motivated by all mentioned works, our current paper investigates the exponential stability of solutions to system (1.1) with a strong damping and a Balakrishnan-Taylor damping. As mentioned at the end of the paper, the Balakrishnan-Taylor damping (alone) is insufficient to deduce exponential stability. For this reason, we add another damping to obtain the uniform stability.
The structure of the paper is as follows. In the next section, we present some fundamental materials that will be used in proving our main results. In the third section, The well-posedness of the problem is demonstrated. We show the exponential stability of system (1.1) in the last section.

# 2 Preliminaries

Here and the sequel, we use $\|\cdot\|$ to denote the usual norm in $L^2(\Omega)$.

We define the space

$$V = \{w \in H^2(\Omega) : w = 0 \text{ on } \{0, \pi\} \times (-d, d)\},$$

with the scalar product

$$(p, q) = \int_\Omega [\Delta p \Delta q + (1-\mu)(2p_{xy}q_{xy} - p_{xx}q_{yy} - p_{yy}q_{xx})] \, dxdy.$$

We note that $(V, (\cdot,\cdot))$ is a Hilbert space, and we have that the norm $\|.\|_V$ is equivalent to the $H^2$ norm (see [10, Lemma 4.1]).

Moreover we consider

$$\mathcal{H}(\Omega) := \text{ The dual space of } V,$$

and we indicate by $\langle .,.\rangle_{2,-2}$ the associated duality.

We have

**Lemma 2.1** *([10]) If $0 < \mu < \frac{1}{2}$ and $f \in L^2(\Omega)$, then there is a unique $p \in V$ such that, for all $q \in V$, we have*

$$(p, q) = \int_\Omega fq. \tag{2.4}$$

The function $p \in V$ satisfying (2.4) is known as the weak solution to the stationary problem

$$\begin{cases} \Delta^2 p = f \\ p(0,y) = p(\pi,y) = p_{xx}(0,y) = p_{xx}(\pi,y) = 0, \\ p_{yy}(x,\pm d) + \mu p_{xx}(x,\pm d) = p_{yyy}(x,\pm d) + (2-\mu)p_{xxy}(x,\pm d) = 0. \end{cases} \tag{2.5}$$

**Lemma 2.2** *([27] Let $p \in V$ and $1 \leq r < +\infty$. Then, we have*

$$\|p\|_r^r \leq C_r \|p\|_V^r, \tag{2.6}$$

*for some positive constant $C_r = C_r(\Omega, r)$.*

The energy related to (1.1) is given as follows

$$\mathcal{E}(t) = \frac{1}{2}\|p_t(t)\|^2 + \frac{1}{2}\|p(t)\|_V^2 - \frac{a}{2}\|p_x(t)\|^2 + \frac{b}{4}\|p_x(t)\|^4, \tag{2.7}$$

which satisfies the following identity

$$\mathcal{E}'(t) = -\alpha\|p_t\|_V^2 - \delta\left(\frac{1}{2}\frac{d}{dt}\|p_x\|^2\right)^2 \leq 0, \tag{2.8}$$

This indicates that the energy decreases with time $t$ and $\mathcal{E}(t) \leq \mathcal{E}(0)$, $\forall\, t \geq 0$.

**Remark 2.3** *We remark that the energy is nonnegative if $a < 0$, and this case is equivalent to a stretched plate. However, this scenario is not applicable to real-world bridges ([11]). When $a > 0$, which is the utmost likely situation for bridges, we have $\mathcal{E}(t) < 0$. This issue can be solved by following some ideas from [[1], section 3]. To do this, we define*

$$\begin{aligned} W &:= \{w \in H^1(\Omega) : w = 0 \text{ on } \{0,\pi\} \times (-d,d)\}, \\ C_*^\infty(\Omega) &:= \{w \in C^\infty(\overline{\Omega}) : \exists\, \varepsilon > 0, w(x,y) = 0 \text{ if } x \in [0,\varepsilon] \cup [\pi-\varepsilon,\pi]\}. \end{aligned}$$

*Endowed with the following norm*

$$||p||_W := \left(\int_\Omega |\nabla p|^2 \, dx\, dy\right)^{1/2}, \tag{2.9}$$

*$W$ is a normed space.*

**Remark 2.4** *([1]) $W$ is defined as the completion of $C_*^\infty(\Omega)$ according to the norm $||\cdot||_W$. It is clear that the embedding $V \hookrightarrow W$ is compact and the optimal embedding constant satisfies*

$$\Lambda_1 := \min_{w\in V} \frac{||w||_V^2}{||w||_W^2}.$$

**Lemma 2.5** *Assume that $0 \le a \le \Lambda_1$, then $\mathcal{E}(t) \ge 0$.*

**Proof.** Using Remark 2.4, we obtain the following inequality

$$||w||_W^2 \le \Lambda_1^{-1} ||w||_V^2, \quad \text{for all } w \in V. \tag{2.10}$$

Since,

$$||p_x||^2 \le \int_\Omega |\nabla p|^2 \, dx \, dy \le \Lambda_1^{-1} \, ||p||_V^2,$$

then we have

$$-\frac{a}{2}||p_x||^2 \ge -\frac{a}{2}\Lambda_1^{-1} \, ||p||_V^2, \quad \forall p \in V,$$

and consequently

$$\frac{1}{2}||p||_V^2 - \frac{a}{2}||p_x||^2 \ge \frac{1}{2}||p||_V^2 \left(1 - a\Lambda_1^{-1}\right).$$

So, if $0 \le a \le \Lambda_1$ we conclude that $\frac{1}{2}||p||_V^2 - \frac{a}{2}||p_x||^2 \ge 0$, and therefore $\mathcal{E}(t) \ge 0$. This agrees with the hypothesis of Theorem 4 in [11]. □

# 3 Well-Posedness

**Definition 3.1** *Let $T$ be a positive number. A function*

$$p \in C([0,T], V) \cap C^1([0,T], L^2(\Omega))$$

*is called a weak solution of (1.1) when*

$$\begin{gathered}\langle p_{tt}, w\rangle_{2,-2} + (p, w) + \int_\Omega \left(-a + b\|p_x\|^2 + \delta\langle p_x, p_{xt}\rangle\right) p_x w_x \, dxdy \\ +\alpha(p_t, w) = 0, \ \forall \ w \in V \\ p(x,y,0) = p_0(x,y), \quad p_t(x,y,0) = p_1(x,y),\end{gathered}$$

*for a.e $t \in [0,T]$.*

**Theorem 3.2** *Suppose that $0 \leq a \leq \Lambda_1$ and let $(p_0, p_1) \in V \times L^2(\Omega)$. Then, the problem* (1.1) *has a unique global weak solution on $[0, T]$, for any $T > 0$.*

**Proof.** We apply the Faedo-Galerkin approach. Consider $\{w_j\}_{j=1}^{\infty}$ a basis of $V$ and let $V_k = span\{w_1, w_2, ..., w_k\}$ be subspace of $V$ with finite dimension, which is spanned by the first $k$ vectors. Let

$$p_0^k(x,y) = \sum_{j=1}^{k} a_j w_j(x,y), \quad p_1^k = \sum_{j=1}^{k} b_j w_j(x,y),$$

so that $p_0^k, p_1^k \in V_k$ and

$$p_0^k \to p_0 \text{ in } V, \quad \text{and } p_1^k \to p_1 \text{ in } L^2(\Omega). \tag{3.11}$$

We are looking for a solution of the form

$$p^k(x,y,t) = \sum_{j=1}^{k} c_j(t) w_j(x,y), \tag{3.12}$$

that solves, in $V_k$,

$$\begin{aligned} &\langle p_{tt}^k, w\rangle_{2,-2} + (p^k, w) + \int_\Omega \left(-a + b\|p_x^k\|^2 + \delta\langle p_x^k, p_{xt}^k\rangle\right) p_x^k w_x \, dxdy \\ &\qquad\qquad\qquad + \alpha(p_t^k, w) = 0, \\ &p^k(x,y,0) = p_0^k(x,y), \quad p_t^k(x,y,0) = p_1^k(x,y). \end{aligned} \tag{3.13}$$

It is easy to check that, for any $k \geq 1$, the above problem (3.13) yields a solution $p^k$ on $[0, t_k)$, where $0 < t_k \leq T$. Now, we multiply (3.13) by $c_j'(t)$ and sum over $j = 1, ..., k$, to obtain

$$\frac{d}{dt}\mathcal{E}^k(t) = -\alpha\|p_t^k\|_V^2 - \delta\left(\frac{1}{2}\frac{d}{dt}\|p_x^k\|^2\right)^2 \leq 0, \tag{3.14}$$

where

$$\mathcal{E}^k(t) = \frac{1}{2}\|p_t^k(t)\|^2 + \frac{1}{2}\|p^k(t)\|_V^2 - \frac{a}{2}\|p_x^k(t)\|^2 + \frac{b}{4}\|p_x^k(t)\|^4. \tag{3.15}$$

Now, we integrate (3.14) over $(0, t)$, where $0 < t < t_k$ and noting, from (3.11), that $(p_0^k)$ and $(p_1^k)$ are respectively bounded in $V$ and $L^2(\Omega)$, we obtain

$$\mathcal{E}^k(t) + \alpha\int_0^t \|p_t^k\|_V^2 ds + \delta\int_0^t \left(\frac{1}{2}\frac{d}{dt}\|p_x^k\|^2\right)^2 ds \leq \mathcal{E}^k(0) \leq C, \tag{3.16}$$

where $C$ is a positive constant that does not depend on $t$ and $k$, and that may varies from line to line.

Hence, we get the bounds

$$\|p^k\|_V^2,\ \|p_t^k(t)\|_{L^2(\Omega)}^2,\ \int_0^t \|p_t^k\|_V^2 \le C. \tag{3.17}$$

As a result, one obtains that $t_k = T$ and we have

$$\begin{cases} (p^k) \quad \textit{is bounded in} \quad L^\infty(0,T;V) \\ (p_t^k) \quad \textit{is bounded in} \quad L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V). \end{cases} \tag{3.18}$$

Hence, there exists a subsequence of $(p^k)$, still denoted by $(p^k)$ verifying

$$\begin{cases} p^k \rightharpoonup p \quad \textit{weakly star in} \quad L^\infty(0,T;V) \\ p_t^k \rightharpoonup p_t \quad \textit{weakly star in} \quad L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V) \\ p^k \longrightarrow p \quad \textit{in } L^2(Q) \ \textit{strongly and a.e} \\ \|p_x^k\|^2 p_{xx}^k \rightharpoonup \mathcal{X}_1 \quad \textit{weakly star in} \quad L^\infty(0,T;L^2(\Omega)) \\ \langle p_x^k, p_{xt}^k\rangle p_{xx}^k \rightharpoonup \mathcal{X}_2 \quad \textit{weakly star in} \quad L^\infty(0,T;L^2(\Omega)), \end{cases} \tag{3.19}$$

where $Q = \Omega \times (0,T)$.

Next, we will prove that $\mathcal{X}_1 = \|p_x\|^2 p_{xx}$ and $\mathcal{X}_2 = \langle p_x, p_{xt}\rangle p_{xx}$ by following the same arguments as in [2, 3]. For the first one, the following lemma is required

**Lemma 3.3** *Suppose that $p, q \in V$. We have*

$$\langle \|p_x\|^2 p_{xx} - \|q_x\|^2 q_{xx}, p - q\rangle \le 0.$$

**Proof.** One has

$$\begin{aligned} &\langle \|p_x\|^2 p_{xx} - \|q_x\|^2 q_{xx}, p - q\rangle \\ &= \|p_x\|^2 \left(\langle p_x, q_x\rangle - \|p_x\|^2\right) + \|q_x\|^2 \left(\langle p_x, q_x\rangle - \|q_x\|^2\right) \\ &\le \|p_x\|^2 \left(\|p_x\|\|q_x\| - \|p_x\|^2\right) + \|q_x\|^2 \left(\|p_x\|\|q_x\| - \|q_x\|^2\right) \\ &= -\left(\|p_x\| - \|q_x\|\right)\left(\|p_x\|^3 - \|q_x\|^3\right) \le 0 \end{aligned}$$

□ Now, let $q \in L^2(0,T;V)$. From Lemma (3.3), one obtains that

$$\int_0^T \langle \|p_x^k\|^2 p_{xx}^k - \|q_x\|^2 q_{xx}, p^k - q\rangle dt \le 0.$$

But

$$\int_0^T \langle \|p_x^k\|^2 p_{xx}^k, p^k\rangle dt = \int_0^T \langle \|p_x^k\|^2 p_{xx}^k, p\rangle dt + \int_0^T \langle \|p_x^k\|^2 p_{xx}^k, p^k - p\rangle dt.$$

Letting $k \to +\infty$, the first integral on the right hand side converges to $\int_0^T \langle \mathcal{X}_1, p\rangle dt$, while the second integral goes to zero since $p^k \to p$ in $L^2(Q)$ strongly.

Since

$$\int_0^T \langle \|q_x\|^2 q_{xx}, p^k - q\rangle dt \longrightarrow \int_0^T \langle \|q_x\|^2 q_{xx}, p - q\rangle dt,$$

we get

$$\int_0^T \langle \mathcal{X}_1 - \|q_x\|^2 q_{xx}, p - q\rangle dt \le 0.$$

Set $q = p - \lambda w$, with $\lambda > 0$ and $w \in L^2(0,T;V)$. Letting $\lambda \to 0^+$, one derives that

$$\int_0^T \langle \mathcal{X}_1 - \|p_x\|^2 p_{xx}, w\rangle dt \le 0.$$

Change $w$ by $-w$ in the last inequality, we deduce that

$$\int_0^T \langle \mathcal{X}_1 - \|p_x\|^2 p_{xx}, w\rangle dt = 0, \ \ \forall\, w \in L^2(0,T;V).$$

Hence, we have

$$\mathcal{X}_1 = \|p_x\|^2 p_{xx}.$$

Next, to prove that $\mathcal{X}_2 = \langle p_x, p_{xt}\rangle p_{xx}$, we note, firstly, that

$$\langle p_x^k, p_{xt}^k\rangle = -\langle p_{xx}, p_t^k\rangle - \langle p^k - p, p_{xxt}^k\rangle.$$

It is clear that $\langle p_{xx}, p_t^k\rangle \longrightarrow \langle p_{xx}, p_t\rangle$ in $L^\infty(0,T)$. Besides, from (3.17), we have

$$\begin{aligned}\int_0^T |\langle p^k - p, p_{xxt}^k\rangle| dt &\le \left(\int_0^T |p^k - p|^2 dt\right)^{\frac{1}{2}} \left(\int_0^T |p_{xxt}^k|^2 dt\right)^{\frac{1}{2}} \\ &\le C\left(\int_0^T |p^k - p|^2 dt\right)^{\frac{1}{2}}\end{aligned}$$

Hence $\langle p^k - p, p_{xxt}^k\rangle \longrightarrow 0$ in $L^1(0,T)$, and so we deduce that

$$\langle p_x^k, p_{xt}^k\rangle \longrightarrow \langle p_x, p_{xt}\rangle \ \text{ in } L^1(0,T). \tag{3.20}$$

Now, let $\varphi \in L^1(0,T;L^2(\Omega))$. Then

$$\begin{aligned}\int_0^T \langle p_x, p_{xt}\rangle \langle p_{xx}, \varphi\rangle dt &= \int_0^T \langle p_x^k, p_{xt}^k\rangle \langle p_{xx}^k, \varphi\rangle dt \\ &+ \int_0^T [\langle p_x, p_{xt}\rangle - \langle p_x^k, p_{xt}^k\rangle] \langle p_{xx}^k, \varphi\rangle dt \\ &+ \int_0^T \langle p_{xx} - p_{xx}^k, \langle p_x, p_{xt}\rangle \varphi\rangle dt\end{aligned} \tag{3.21}$$

The last two terms on the right side of (3.21) go to zero as $k \to +\infty$ by (3.17) and (3.20). Since $\varphi$ is arbitrary, we conclude that $\mathcal{X}_2 = \langle p_x, p_{xt}\rangle p_{xx}$.

Now, integrate (3.13) over $(0,t)$ to obtain

$$\int_\Omega p_t^k w \, dxdy + \int_0^t (p^k, w) ds + \int_0^t \int_\Omega \left(-a + b\|p_x^k\|^2 + \delta\langle p_x^k, p_{xt}^k\rangle\right) p_x^k w_x \, dxdyds$$
$$+\alpha \int_0^t (p_t^k, w) ds = \int_\Omega p_1^k w, \ \forall\, w \in V. \tag{3.22}$$

Letting $k \to \infty$, we get

$$\int_\Omega p_t w \, dxdy - \int_\Omega p_1 w \ = \ -\int_0^t (p, w) ds - \int_0^t \int_\Omega \left(-a + b\|p_x\|^2 + \delta\langle p_x, p_{xt}\rangle\right) p_x w_x \, dxdyds$$
$$- \int_0^t \alpha(p_t, w) ds. \tag{3.23}$$

This means that (3.23) holds true for any $w \in V$. Since the terms in the right hand side of (3.23) are absolutely continuous, then (3.23) is differentiable for a.e. $t \geq 0$. It holds that

$$\langle p_{tt}, w\rangle_{2,-2} + (p, w) + \int_\Omega \left(-a + b\|p_x\|^2 + \delta\langle p_x, p_{xt}\rangle\right) p_x w_x \, dxdy$$
$$+\alpha(p_t, w) = 0, \ \forall\, w \in V. \tag{3.24}$$

Regarding the initial conditions, from (3.19) and using Lions's Lemma [21], we can simply get

$$p^k \to p \quad in \ \ C([0,T), L^2(\Omega)). \tag{3.25}$$

$p^k(x, y, 0)$ then makes sense and $p^k(x, y, 0) \to p(x, y, 0)$ in $L^2(\Omega)$. Noting that

$$p^k(x, y, 0) = p_0^k(x, y) \to p_0(x, y) \quad in \ \ V,$$

we get

$$p(x, y, 0) = p_0(x, y). \tag{3.26}$$

Besides, as in [19], we multiply (3.13) by $\phi \in C_0^\infty(0,T)$ and integrate over $(0,T)$, we get, for any $j \leq k$,

$$-\int_0^T \int_\Omega p_t^k(t) w \phi'(t) \, dx \, dy \, dt$$
$$= -\int_0^T (p^k, w)\phi(t) dt - \int_0^T \int_\Omega \left(-a + b\|p_x^k\|^2 + \delta\langle p_x^k, p_{xt}^k\rangle\right) p_x^k w_x \phi(t) \, dx \, dy \, dt$$
$$-\alpha \int_0^T (p_t^k, w)\phi(t) dt. \tag{3.27}$$

As $k \to +\infty$, we have for any $w \in V$ and any $\phi \in C_0^\infty(0,T)$

$$\begin{aligned}
&-\int_0^T \int_\Omega p_t(t) w \phi'(t)\, dx\, dy\, dt \\
&= -\int_0^T (p,w)\phi(t) dt - \int_0^T \int_\Omega \left(-a + b\|p_x\|^2 + \delta\langle p_x, p_{xt}\rangle\right) p_x w_x \phi(t)\, dx\, dy\, dt \\
&-\alpha \int_0^T (p_t, w)\phi(t) dt, \qquad (3.28)
\end{aligned}$$

which implies that (see [19]),

$$p_{tt} \in L^2(0,T;\mathcal{H}(\Omega)).$$

As $p_t \in L^2(0,T;L^2(\Omega))$, we conclude that $p_t \in C(0,T;\mathcal{H}(\Omega))$.
$p_t^k(x,y,0)$ therefore makes sense and

$$p_t^k(x,y,0) \to p_t(x,y,0) \ \ in \ \ \mathcal{H}(\Omega).$$

However

$$p_t^k(x,y,0) = p_1^k(x,y) \to p_1(x,y) \ \ in \ \ L^2(\Omega).$$

So,

$$p_t(x,y,0) = p_1(x,y) \qquad (3.29)$$

For the uniqueness, Assume that $p$ and $\overline{p}$ verify (3.24), (3.26) and (3.29). So, by integrating by parts, $q = p - \overline{p}$ satisfy

$$\begin{aligned}
&\int_\Omega q_{tt}(x,t) w\, dx\, dy + (q,w) + a\int_\Omega q_{xx} w\, dx\, dy - b\int_\Omega \left(\|p_x\|^2 p_{xx} - \|\overline{p}_x\|^2 \overline{p}_{xx}\right) w\, dx\, dy \\
&\quad - \int_\Omega \delta\left(\langle p_x, p_{xt}\rangle p_{xx} - \langle \overline{p}_x, \overline{p}_{xt}\rangle \overline{p}_{xx}\right) w\, dx\, dy + \alpha(q_t, w) = 0,\ \forall\, w \in V, \qquad (3.30) \\
&\quad q(x,y,0) = q_t(x,y,0) = 0.
\end{aligned}$$

(3.30) holds true for any $w \in C_0^\infty(\Omega \times (0,T))$, by density it is also valid for any $w \in L^2(\Omega \times (0,T))$.

If we change $w$ by $q_t$ in (3.30), we get

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\|q_t\|^2 + \frac{1}{2}\|q\|_V^2\right\} + a\int_\Omega q_{xx} q_t\, dx\, dy - b\int_\Omega \left(\|p_x\|^2 p_{xx} - \|\overline{p}_x\|^2 \overline{p}_{xx}\right) q_t\, dx\, dy \\
&\quad - \int_\Omega \delta\left(\langle p_x, p_{xt}\rangle p_{xx} - \langle \overline{p}_x, \overline{p}_{xt}\rangle \overline{p}_{xx}\right) q_t\, dx\, dy + \alpha\|q_t\|_V^2 = 0 \qquad (3.31)
\end{aligned}$$

By using Young's inequality, we get

$$\begin{aligned}
-a\int_\Omega q_{xx} q_t\, dx\, dy &\leq \frac{a}{2}\|q_t\|^2 + \frac{a}{2}\|q_{xx}\|^2 \\
&\leq C\left(\|q_t\|^2 + \|q\|_V^2\right) \qquad (3.32)
\end{aligned}$$

Next, it is easy to see that

$$\begin{aligned} &\|p_x\|^2 p_{xx} - \|\overline{p}_x\|^2 \overline{p}_{xx} \\ &= \|p_x\|^2 p_{xx} - \langle p + \overline{p}, q_{xx} \rangle \overline{p}_{xx}, \end{aligned} \tag{3.33}$$

and

$$\begin{aligned} &\int_\Omega \delta \left( \langle p_x, p_{xt} \rangle p_{xx} - \langle \overline{p}_x, \overline{p}_{xt} \rangle \overline{p}_{xx} \right) q_t \, dx \, dy \\ &= -\delta (\langle q_t, \overline{p}_{xx} \rangle)^2 - \delta \langle q_t, \overline{p}_{xx} \rangle \langle p_t, q_{xx} \rangle - \delta \langle p_t, p_{xx} \rangle \langle q_t, q_{xx} \rangle \end{aligned} \tag{3.34}$$

Then, using (3.33), we infer that

$$\begin{aligned} &b \int_\Omega \left( \|p_x\|^2 p_{xx} - \|\overline{p}_x\|^2 \overline{p}_{xx} \right) q_t \, dx \, dy \\ &\leq C \|q_{xx}\| \|q_t\| \\ &\leq C \left( \|q_t\|^2 + \|q\|_V^2 \right) \end{aligned} \tag{3.35}$$

Besides, from (3.34), one derives

$$\begin{aligned} &\int_\Omega \delta \left( \langle p_x, p_{xt} \rangle p_{xx} - \langle \overline{p}_x, \overline{p}_{xt} \rangle \overline{p}_{xx} \right) q_t \, dx \, dy \\ &\leq -\delta (\langle q_t, \overline{p}_{xx} \rangle)^2 + C \|q_{xx}\| \|q_t\| \\ &\leq -\delta (\langle q_t, \overline{p}_{xx} \rangle)^2 + C \left( \|q_t\|^2 + \|q\|_V^2 \right) \end{aligned} \tag{3.36}$$

By using (3.32), (3.35) and (3.36) we can deduce that

$$\begin{aligned} &\frac{d}{dt} \left\{ \frac{1}{2} \|q_t\|_{L^2(\Omega)}^2 + \frac{1}{2} \|q\|_V^2 \right\} + \delta (\langle q_t, \overline{p}_{xx} \rangle)^2 + \alpha \|q_t\|_V^2 \\ &\leq C \left( \|q_t\|^2 + \|q\|_V^2 \right) \end{aligned} \tag{3.37}$$

By Gronwall's inequality, we obtain

$$\|q_t\|^2 + \|q\|_V^2 \quad \leq \quad C e^{Ct} \left( \|q_t(0)\|^2 + \|q(0)\|_V^2 \right),$$

which gives that $q = 0$ and thus $p = \overline{p}$. □

The following theorem gives an additional regularity result.

**Theorem 3.4** *Suppose that $0 \leq a \leq \Lambda_1$ holds true and let $(p_0, p_1) \in X \times V$, with $X = H^4(\Omega) \cap V$. Then there is a unique function $p = p(x, y, t)$ fulfilling the initial conditions (3.26) and (3.29), and that satisfies*

$$p \in L^\infty(0, T; X), \; p_t \in L^\infty(0, T; V) \cap L^2(0, T; X), \quad \textit{and} \;\; p_{tt} \in L^2(0, T; L^2(\Omega)),$$

*and*

$$p_{tt} + \Delta^2 p - \left( \phi(p) + \delta \langle p_x, p_{xt} \rangle \right) p_{xx} + \alpha \Delta^2 p_t = 0, \quad \textit{in } L^2(0, T; L^2(\Omega)). \tag{3.38}$$

**Proof.** Taking $\{w_j\}_{j=1}^{\infty}$ as a basis of $X$. The solutions $p^k$ can be written in the form (3.12) and verify (3.13) as well as the following initial conditions

$$p_0^k \to p_0 \ \ in\ X, \quad and\ p_1^k \to p_1 \ \ in\ V.$$

It is easy to see that the bounds (3.17) are satisfied. In (3.13), let $w = \Delta^2 p_t^k$ and by integration by parts we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|p_t^k\|_V^2 + \|\Delta^2 p^k\|^2\right) + \alpha\|\Delta^2 p^k\|^2 \;&=\; -a\langle p_{xx}^k, \Delta^2 p_t^k\rangle + b\|p_x^k\|^2\langle p_{xx}^k, \Delta^2 p_t^k\rangle \\ &+\; \delta\langle p_x^k, p_{xt}^k\rangle\langle p_{xx}^k, \Delta^2 p^k\rangle\end{aligned}$$

Therefore, using (3.17), it follows that

$$\begin{aligned}&\|p_t^k\|_V^2 + \|\Delta^2 p^k\|^2 + 2\alpha\int_0^t \|\Delta^2 p^k\|^2 ds \\ &\le C_1 + C_2\int_0^t |\langle p_{xx}^k, \Delta^2 p_t^k\rangle| ds \\ &\le C_1 + C_2\int_0^t |p_{xx}^k|^2 ds + \alpha\int_0^t |\Delta^2 p_t^k|^2 ds\end{aligned}$$

Hence,

$$\|p_t^k\|_V^2, \ \ \|\Delta^2 p^k\|^2, \quad and \quad \int_0^t |\Delta^2 p_t^k|^2 ds \le C.$$

We proceed as in Theorem (3.2) to prove the existence of a unique $p \in L^\infty(0,T;X)$ satisfying (3.38), (3.26), (3.29) and $p_t \in L^\infty(0,T;V)\cap L^2(0,T;X)$. It follows from (3.38) that $p_t \in L^2(0,T;L^2(\Omega))$. □

# 4 Exponential stability

This section's major result is the following:

**Theorem 4.1** *Let $0 \le a \le \gamma\Lambda_1$, where $0 < \gamma < 1$. Then, there are two constants $K > 0$ and $\lambda > 0$ such that the energy defined in (2.7) verifies*

$$\mathcal{E}(t) \le K e^{-\lambda t}, \ \ \forall\, t \ge 0. \tag{4.39}$$

**Proof.** We will work with regular solutions and by standard density arguments, the decay holds true even for weak solutions. Multiplying (1.1) by $p$ and integrating over $\Omega \times (s,T)$, for $0 < s < T$, we get

$$\int_s^T \int_\Omega \left(p_{tt}p + p\Delta^2 p - \phi(p)p_{xx}p - \delta\langle p_x, p_{xt}\rangle p_{xx}p + \alpha p\Delta^2 p_t\right)\, dxdydt = 0, \tag{4.40}$$

Using Lemma (2.1) and integration by parts to obtain

$$\int_s^T \int_\Omega (p_t p)_t \, dxdydt - \int_s^T \int_\Omega p_t^2 \, dxdydt + \int_s^T \|p\|_V^2 \, dt + \int_s^T \int_\Omega \phi(p) p_x^2 \, dxdydt$$
$$+ \int_s^T \int_\Omega \delta \langle p_x, p_{xt} \rangle p_x^2 \, dxdydt + \alpha \int_s^T (p, p_t) \, dt = 0 \tag{4.41}$$

This yields

$$\int_s^T \mathcal{E}(t) \, dt + \int_s^T \int_\Omega (p_t p)_t - \frac{3}{2} \int_s^T \int_\Omega p_t^2 + \frac{1}{2} \int_s^T \|p\|_V^2 - \frac{a}{2} \int_s^T \|p_x\|^2$$
$$+ \frac{3b}{4} \int_s^T \|p_x\|^4 + \delta \int_s^T \int_\Omega \langle p_x, p_{xt} \rangle p_x^2 \, dxdydt + \alpha \int_s^T (p, p_t) \, dt = 0 \tag{4.42}$$

Then, we obtain

$$\begin{aligned} \int_s^T \mathcal{E}(t) \, dt \quad &\leq \quad - \int_s^T \int_\Omega (p_t p)_t + \frac{3}{2} \int_s^T \int_\Omega p_t^2 + \frac{a}{2} \int_s^T \|p_x\|^2 \\ &\quad - \quad \delta \int_s^T \int_\Omega \langle p_x, p_{xt} \rangle p_x^2 \, dxdt - \alpha \int_s^T (p, p_t) \, dt \end{aligned} \tag{4.43}$$

The terms on the right hand side of (4.43) can be estimated as follows. Using Lemma (2.2) and Young's inequality, we infer that

$$\begin{aligned} \left| - \int_s^T \int_\Omega (p_t p)_t \right| \quad &\leq \quad \left| \int_\Omega p_t(s) p(s) \right| + \left| \int_\Omega p_t(T) p(T) \right| \\ &\leq \quad \frac{1}{2} \int_\Omega p_t^2(s) + \frac{1}{2} \int_\Omega p_t^2(T) + \frac{1}{2} \int_\Omega p^2(s) + \frac{1}{2} \int_\Omega p^2(T) \\ &\leq \quad \mathcal{E}(s) + \mathcal{E}(T) + C\|p(s)\|_V^2 + C\|p(T)\|_V^2 \\ &\leq \quad C\mathcal{E}(s), \end{aligned} \tag{4.44}$$

where $C$ is a generic positive constant. For the second term, thanks to Lemma (2.2) we have

$$\frac{3}{2} \int_s^T \int_\Omega p_t^2 \leq C \int_s^T \|p_t\|_V^2 = \frac{C}{\alpha} \int_s^T (-\mathcal{E}'(t)) \, dt \leq \frac{C}{\alpha} \mathcal{E}(s). \tag{4.45}$$

The third term on the right hand side of (4.43) may be estimated as

$$\frac{a}{2} \int_s^T \|p_x\|^2 \leq \frac{a\Lambda_1^{-1}}{2} \int_s^T \|p\|_V^2 \leq \gamma \int_s^T \mathcal{E}(t) dt \tag{4.46}$$

Thanks to Young's inequality, we deduce, for any $\varepsilon > 0$, that

$$\begin{aligned} \left| - \delta \int_s^T \int_\Omega \langle p_x, p_{xt} \rangle p_x^2 \, dxdt \right| \quad &\leq \quad C_\varepsilon \int_s^T \left( \frac{d}{dt} \|p_x\|^2 \right)^2 dt + \frac{\delta\varepsilon}{4} \int_s^T \|p_x\|^4 dt \\ &\leq \quad \leq C_\varepsilon \int_s^T (-\mathcal{E}'(t)) \, dt + \frac{\delta\varepsilon}{b} \int_s^T \mathcal{E}(t) dt \\ &\leq \quad C_\varepsilon \mathcal{E}(s) + \frac{\delta\varepsilon}{b} \int_s^T \mathcal{E}(t) dt, \end{aligned} \tag{4.47}$$

and

$$\begin{aligned}\left|-\alpha \int_s^T (p, p_t)\, dt\right| &\leq \alpha \int_s^T \|p\|_V \|p_t\|_V \\ &\leq C_\varepsilon \int_s^T \|p_t\|_V^2 dt + \frac{\alpha\varepsilon}{2} \int_s^T \|p\|_V^2 dt \\ &\leq C_\varepsilon \mathcal{E}(s) + \alpha\varepsilon \int_s^T \mathcal{E}(t) dt. \end{aligned} \tag{4.48}$$

By inserting (4.44), (4.45), (4.46), (4.47) and (4.48) into (4.43), and choosing $\varepsilon$ such that $1 - \gamma - (\frac{\delta}{b} + \alpha)\varepsilon > 0$, we coclude that there exists a positive constant $C_1$ satisfying

$$\int_s^T \mathcal{E}(t)\, dt \leq C_1 \mathcal{E}(s),\ \forall\, s > 0.$$

By letting $T \to +\infty$, and thanks to Theorem 8.1 in [18], we get the desired inequality (4.39). □

**Remark 4.2** *As remarked in [28] (for extensible beams), The Balakrishnan-Taylor damping does not seem sufficient to provide "a good" stability result to our problem. In fact, if $a = b = \alpha = 0$ in (1.1), we have the following equation*

$$p_{tt} + \Delta^2 p - \delta\langle p_x, p_{xt}\rangle p_{xx} = 0, \quad \text{in } \Omega \times (0, +\infty). \tag{4.49}$$

*The corresponding energy for system (4.49) is*

$$\mathcal{E}(t) = \frac{1}{2}\|p_t\|^2 + \frac{1}{2}\|p\|_V^2,$$

*which satisfies*

$$\mathcal{E}'(t) = -\delta[\langle p_x, p_{xt}\rangle]^2 \leq 0,\ \forall\, t > 0. \tag{4.50}$$

*Hence, the system is dissipative. One can ask about its stability. If the system is strongly stable, that is, $\mathcal{E}(t) \to 0$ as $t \to +\infty$, then we get*

$$|\langle p_x, p_{xt}\rangle| \leq C\mathcal{E}(t) \to 0, \ \text{ as } t \to +\infty. \tag{4.51}$$

*This indicates that the Balakrishnan-Taylor damping gets less and less effective as $t \to +\infty$. In addition, it is clear, from (4.50)-(4.51), that*

$$\begin{aligned}\mathcal{E}(t) &\geq \frac{|\langle p_x, p_{xt}\rangle|}{C} \\ &\geq \frac{\sqrt{-\mathcal{E}'(t)}}{C\sqrt{\delta}}\end{aligned}$$

*This latter gives us*

$$-\mathcal{E}'(t)\mathcal{E}^{-2}(t) \leq C^2\delta \tag{4.52}$$

*Integrating inequatity* (4.52) *over* $(0, t)$, *it follows that*

$$\mathcal{E}(t) \geq \frac{1}{\delta C^2 t + \mathcal{E}(0)^{-1}},\ t > 0, \tag{4.53}$$

*which means that the energy is bounded from below polynomially, and consequently the Balakrishnan-Taylor damping term* $-\delta\langle p_x, p_{xt}\rangle p_{xx}$ *(alone) is no longer enough to ensure exponential stability. In conclusion, we need to add another damping term, like a strong damping of the form* $\alpha\Delta^2 p_t$, *to recover exponential decay for system* (4.49).

# Conclusion

This paper focuses on the existence and the exponential stability of solutions for plate equation subject to a Balakrishnan-Taylor damping and a strong one. This equation models the deformation of the deck of a suspension bridge. This work is motivated by previous results concerning the stability of a suspension bridge [5, 6, 17, 22, 24] and some other problems subject to a Balakrishnan-Taylor damping [7, 14, 28, 29].
As a future works, we can change the type of damping by considering, for example, structural damping (of the form $\Delta y_t$). Also, we can study a coupled Balakrishnan-Taylor plate with only one strong damping.

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# Conflict of interest

The author declares there is no conflicts of interest.

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