Research article

The inverses of tails of the generalized Riemann zeta function within the range of integers

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Abstract: In recent years, many mathematicians researched infinite reciprocal sums of various of sequences and evaluated their value by the asymptotic formulas. We study the asymptotic formulas of the infinite reciprocal sums formed as \((\sum_{k=n}^{\infty} \frac{1}{k(k+s)^r})^{-1}\) for \(r, s, t \in \mathbb{N}^+\), where the asymptotic formulas are polynomials.

Keywords: Riemann zeta function; Hurwitz zeta function; Reciprocal sums; Asymptotic formulas; Infinite sums

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1. Introduction

In past years, many mathematicians have been working on partial infinite sums of reciprocal linear recurrence sequences.

In 2008, Ohtsuka Hideyuki and Nakamura Shigeru [11] derived the formulas of the partial infinite sums of reciprocal Fibonacci numbers and showed that

\[
\left(\sum_{k=n}^{\infty} \frac{1}{F_k}\right)^{-1} = \begin{cases} 
F_{n-2}, & \text{if } n \text{ is even, } n \geq 2; \\
F_{n-2} - 1, & \text{if } n \text{ is odd, } n \geq 1,
\end{cases}
\]  

(1.1)

\[
\left(\sum_{k=n}^{\infty} \frac{1}{F_k^2}\right)^{-1} = \begin{cases} 
F_{n-1}F_n - 1, & \text{if } n \text{ is even, } n \geq 2; \\
F_{n-1}F_n, & \text{if } n \text{ is odd, } n \geq 1.
\end{cases}
\]  

(1.2)

They only given the value of the integer part, there existed other formulas with smaller error, for example, Takao Komatsu [4] researched the nearest integer of the sum of reciprocal Fibonacci numbers
and derived
\[
\left\| \left( \sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\| = F_n - F_{n-1},
\]
(1.3)
\[
\left\| \left( \sum_{k=n}^{\infty} \frac{(-1)^n}{F_k} \right)^{-1} \right\| = (-1)^n(F_n + F_{n+1}),
\]
(1.4)
where \(\|\cdot\|\) denoted the nearest integer, in other words, \(\|x\| = \lfloor x - \frac{1}{2} \rfloor\).

In 2020, Ho-Hyeong Lee and Jong-Do Park [5] gave the concept of asymptotic formulas which were more accurate, the conclusions were as follows:
\[
\left\| \left( \sum_{k=n}^{\infty} \frac{1}{F_{k+k+2l}} \right)^{-1} \right\| \sim F_{n+l-1}F_{n+l} - (F_l^2 + (-1)^l) \frac{(-1)^n}{3},
\]
(1.5)
\[
\left\| \left( \sum_{k=n}^{\infty} \frac{1}{F_{k+k+2l-1}} \right)^{-1} \right\| \sim F_{n+l-1}^2 - (F_{l-1}F_l + (-1)^l) \frac{(-1)^n}{3},
\]
where \(a_n \sim b_n\) meant \(\lim_{n \to \infty} (a_n - b_n) = 0\). For more other results related to the infinite reciprocal sums of linear recurrence sequences, see [1, 10, 13] and references therein.

The zeta function \(\zeta(z)\) is undoubtedly the most famous function in analytic number theory, initially studied by Euler and achieved prominence with Riemann, which had been abstracting the attention of many mathematicians. Indeed, Another well-known sequence harmonic number \(H_n\) is sum of the first \(n\) terms of \(\zeta(z)\) when \(z = 1\), and the generating function of harmonic numbers \(\sum_{n=1}^{\infty} H_n x^n\) is an important tool to study property of \(H_n\). Kim [14, 15, 16, 17] derive many worthy and interesting results associated with the zeta function, harmonic number and its generating function, which inspires us deeply.

As the same time, many researchers began to study the tails of well-known functions such as Riemann zeta function and Hurwitz zeta function in [2, 3, 6, 9, 12].

For example, Kim Donggyun and Song Kyunghwan [3] studied the inverses of tails of the Riemann zeta function and derived for \(s\) on the critical strip \(0 < s < 1\),
\[
\left\| \left( \sum_{k=n}^{\infty} \frac{1}{k^s} \right)^{-1} \right\| \sim \begin{cases} 2(1 - 2^{1-s})(n - \frac{1}{2})^s, & \text{if } n \text{ is even;} \\ -2(1 - 2^{1-s})(n - \frac{1}{2})^s, & \text{if } n \text{ is odd.} \end{cases}
\]
(1.6)

Ho-Hyeong Lee and Jong-Do Park [6] dealt with the inverses of tails of Hurwitz zeta function when \(s \geq 2, s \in \mathbb{N}\) and \(0 \leq a < 1\), and derived
\[
\left( \sum_{k=n}^{\infty} \frac{1}{(k+a)^s} \right)^{-1} \sim \sum_{j=0}^{s-1} A_j(j + a)^j,
\]
(1.7)
where \(A_{j-1} = s - 1, A_j = -\sum_{j=1}^{s-1} x_j A_{j-1}, x_j = (s-2+j)B_j\) and \(B_j\) is Bernoulli numbers.

In this paper, we extend their asymptotic formulas for the methods and results by considering the tails of \(\left( \sum_{k=n}^{\infty} \frac{1}{k(k+1)} \right)^{-1}, \left( \sum_{k=n}^{\infty} \frac{1}{k(k+1)^2} \right)^{-1}\) and \(\left( \sum_{k=n}^{\infty} \frac{1}{k(k+1)^3} \right)^{-1}\). Further revealing the property of reciprocal sums of the various sequences.
2. Main results

Before our conclusion, we define \( \binom{t}{i} = 0 \) for all \( i \in \mathbb{N}^+ \), which will take effect in expressing the asymptotic formulas in Theorem 3.

**Theorem 1**: For all \( m \in \mathbb{N} \), we have

\[
\left( \sum_{k=n}^{\infty} \frac{1}{k(k+t)} \right)^{-1} \sim n + \frac{t}{2} - \frac{1}{2}.
\]

**Theorem 2**: For all \( m \in \mathbb{N} \), we have

\[
\left( \sum_{k=n}^{\infty} \frac{1}{[k(k+t)]^2} \right)^{-1} \sim 3n^3 + an^2 + bn + c,
\]

where

\[
a = \frac{9}{2} t - \frac{9}{2},
\]

\[
b = \frac{27}{20} t^2 - \frac{9}{2} t + \frac{15}{4},
\]

\[
c = -\frac{3}{40} t^3 - \frac{27}{40} t^2 + \frac{15}{8} t - \frac{9}{8}.
\]

**Theorem 3**: If \( r + s - 1 > 0 \) and \( r, s, t \in \mathbb{N} \), then there exists unique polynomial

\[
B(n, r, s, t) = b_{r+s-1}n^{r+s-1} + b_{r+s-2}n^{r+s-2} + \cdots + b_1n + b_0,
\]

subject to

\[
\left( \sum_{k=n}^{\infty} \frac{1}{n'(n+t)^s} \right)^{-1} \sim B(n, r, s, t),
\]

where

\[
b_{r+s-1} = r + s - 1,
\]

\[
b_{r+s-2} = \frac{(r + s - 1)^2}{2(r + s)} \left( 2st - (r + s) \right),
\]

\[
b_{r+s-3} = \frac{(r + s - 1)^2}{12(r + s)^2(r + s + 1)} \left[ 6s(r^2s - r^2 + 2rs^2 - 2rs + s^3 - s^2 - 2s)t^2 \\
- 6s(r + 3r^2s - r^2 + 3rs^2 - 2rs - 2r + s^3 - s^2 - 2s)t \\
+ (2r^4 + 8r^3s - r^3 + 12r^2s^2 - 3r^2s - 3r^2 + 8s^3 - 3rs^2 - 6rs + 2s^4 - s^3 - 3s^2) \right],
\]

\[
\ldots 
\]

\[
\ldots 
\]

\[
b_{i-r-s+1} = \frac{\sum_{k_1+k_2=i}^{s} \binom{s}{k_1} t^{r+s-k_2} \sum_{j=k_1+1}^{r+s-1} b_j f(k_1)}{3r + 3s - i - 3} - \frac{\sum_{k_1+k_2=i}^{s} b_{k_1} \sum_{j=k_2}^{r+s-1} b_j f(k_2)}{3r + 3s - i - 3},
\]
We need several lemmas for the proof. Theorem 3, we derive that coefficients \( b_j \) (0 \( \leq j \leq r + s - 1 \)) are determined by \( r, s \) and \( t \). As the same time, if we calculate \( b_0 \) by using the representation of \( b_i \), there will appear \((r+s-1),(r+s-2), \ldots , (1)\). In order to make \( b_0 \) satisfies the representation of \( b_j \), we the define \( \binom{i}{-1} = 0 \) for \( i \in \mathbb{N}^+ \) in this paper. Obviously, it satisfies \( \binom{n+1}{0} = \binom{n}{0} + \binom{n}{-1} \) for \( n \in \mathbb{N}^+ \).

**Collary 1:** If \( s = 1, m = 0 \), we have

\[
\left( \sum_{k=n}^{\infty} \frac{1}{n^2} \right)^{-1} \sim n - \frac{1}{2}.
\]

**Collary 2:** If \( s = 2, t = 0 \), we have

\[
\left( \sum_{k=n}^{\infty} \frac{1}{n^4} \right)^{-1} \sim 3n^3 - \frac{9}{2}n^2 + \frac{15}{4}n - \frac{9}{8}.
\]

**Collary 3:** If \( s = 2 \), we have

\[
\left( \sum_{k=n}^{\infty} \frac{1}{n^2(n+1)^2} \right)^{-1} \sim 3n^3 + \frac{3}{5}n,
\]

\[
\left( \sum_{k=n}^{\infty} \frac{1}{n^2(n+2)^2} \right)^{-1} \sim 3n^3 + \frac{9}{2}n^2 + \frac{3}{20}n - \frac{5}{8},
\]

\[
\left( \sum_{k=n}^{\infty} \frac{1}{n^2(n+3)^2} \right)^{-1} \sim 3n^3 + 9n^2 + \frac{12}{5}n - \frac{18}{5}.
\]

3. **Proof of Theorem**

3.1. **Proof of Theorem 1**

We need several lemmas for the proof. Lemma 1: Let \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) be sequences of positive real number with \( \lim_{m \to \infty} a_m = \lim_{m \to \infty} b_m = 0 \). If \( a_n < b_n + a_{n+1} \) hold for any \( n \in \mathbb{N}^+ \), then we have

\[
a_n \leq \sum_{k=n}^{\infty} b_k \quad \text{for} \quad n \in \mathbb{N}^+.
\]
**Proof:** See lemma 2.1 in [7]. □

**Lemma 2:** For all \( t \geq 2, \, t \in \mathbb{N}, \) we have

\[
\frac{1}{n + \frac{t}{2} - \frac{1}{2}} < \frac{1}{n(n + t)} + \frac{1}{n + \frac{t}{2} + \frac{1}{2}}.
\]

**Proof:** It is equivalent with

\[
\frac{1}{n + \frac{t}{2} - \frac{1}{2}} - \frac{1}{n(n + t)} - \frac{1}{n + \frac{t}{2} + \frac{1}{2}} < 0. \tag{3.1}
\]

The left side

\[
= \frac{n(n + t) - (n + \frac{t}{2} - \frac{1}{2})(n + \frac{t}{2} + \frac{1}{2})}{(n + \frac{t}{2} - \frac{1}{2})(n + \frac{t}{2} + \frac{1}{2})n(n + t)}
\]

\[
= \frac{-\frac{t^2}{4} + \frac{1}{4}}{(n + \frac{t}{2} - \frac{1}{2})(n + \frac{t}{2} + \frac{1}{2})n(n + t)}.
\]

Hence, we have

\[
-\frac{t^2}{4} + \frac{1}{4} < 0 \quad \text{for} \, n > 2,
\]

so

\[
\frac{1}{n + \frac{t}{2} - \frac{1}{2}} < \frac{1}{n(n + t)} + \frac{1}{n + \frac{t}{2} + \frac{1}{2}} \quad \text{for} \, n > 2.
\]

This completes the proof. □

**Lemma 3:** For all \( \varepsilon > 0, \) there exists \( N_0 > 2, \) subject to

\[
\frac{1}{n + \frac{t}{2} - \frac{1}{2} - \varepsilon} > \frac{1}{n + \frac{t}{2} + \frac{1}{2} - \varepsilon} + \frac{1}{n(n + t)} \quad \text{for} \, n > N_0.
\]

**Proof:** It is equivalent with

\[
\frac{1}{n + \frac{t}{2} - \frac{1}{2} - \varepsilon} - \frac{1}{n + \frac{t}{2} + \frac{1}{2} - \varepsilon} - \frac{1}{n(n + t)} > 0. \tag{3.2}
\]

The left side

\[
= \frac{n^2 + tn - (n^2 + tn + \frac{t^2}{4}) - \frac{1}{2} - 2\varepsilon(n + \frac{t}{2}) + \varepsilon^2}{(n + \frac{t}{2} - \frac{1}{2} - \varepsilon)(n + \frac{t}{2} + \frac{1}{2} - \varepsilon)n(n + t)}
\]

\[
= \frac{2\varepsilon n + t\varepsilon + \frac{1}{4} - \frac{t^2}{4} - \varepsilon^2}{(n + \frac{t}{2} - \frac{1}{2} - \varepsilon)(n + \frac{t}{2} + \frac{1}{2} - \varepsilon)n(n + t)}.
\]

We can restrict \( \varepsilon < 1 \) and fix \( t, \) then we have

\[
t\varepsilon + \frac{1}{4} - \frac{t^2}{4} - \varepsilon^2 = O(1),
\]

so there exists \( N_0 > 2, \) subject to

\[
2\varepsilon n + O(1) > 0 \quad \text{for} \, n > N_0,
\]
then
\[ \frac{1}{n + \frac{t}{2} - \frac{1}{2} - \varepsilon} > \frac{1}{n + \frac{t}{2} + \frac{1}{2} - \varepsilon} + \frac{1}{n(n+t)}, \]
which proves equation (3.2) and completes the proof. \(\square\)

The proof of Theorem 1:

Case 1: When \( t \geq 2 \).

By Lemma 1, Lemma 2 and Lemma 3, we have for all \( \varepsilon > 0 \), there exists \( N_0 > 2 \), subject to
\[ n + \frac{t}{2} - \frac{1}{2} - \varepsilon < \left( \sum_{k=n}^{\infty} \frac{1}{k(n+k)} \right)^{-1} \]
for \( n > N_0 \), (3.3)
hence
\[ \left| \left( \sum_{k=n}^{\infty} \frac{1}{k(k+1)} \right)^{-1} - (n + \frac{t}{2} - \frac{1}{2}) \right| < \varepsilon, \]
in other words,
\[ \left( \sum_{k=n}^{\infty} \frac{1}{k(k+1)} \right)^{-1} \sim (n + \frac{t}{2} - \frac{1}{2}). \]

Case 2: When \( t = 1 \).

\[ \sum_{k=n}^{\infty} \frac{1}{k(k+1)} = \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)} + \cdots \]
\[ = \left( \frac{1}{n} - \frac{1}{n+1} \right) + \left( \frac{1}{n+1} - \frac{1}{n+2} \right) + \left( \frac{1}{n+2} - \frac{1}{n+3} \right) + \cdots \]
\[ = \frac{1}{n}, \]
hence
\[ \left( \sum_{k=n}^{\infty} \frac{1}{k(k+1)} \right)^{-1} \sim n. \]

Case 3: When \( t = 0 \), the proof is similar with the Case 1, we can easily deduce the result. \(\square\)

3.2. Proof of Theorem 2

Lemma 4: Let \( f(n,t,\varepsilon) = 3n^3 + an^2 + bn + c \), and \( a, b, c \) are defined in Theorem 2, then for all \( \varepsilon > 0 \), there exists \( N_1 > 0 \), subject to
\[ \frac{1}{f(n,t,\varepsilon)} < \frac{1}{f(n+1,t,\varepsilon)} + \frac{1}{n^2(n+t)^2} \]
for \( n > N_1 \).

Proof: It is equivalent with
\[ \frac{1}{f(n,t,\varepsilon)} - \frac{1}{f(n+1,t,\varepsilon)} - \frac{1}{n^2(n+t)^2} < 0 \]
for \( n > N_1 \). (3.4)
The left side = \[
\frac{[f(n+1,t,\varepsilon) - f(n,t,\varepsilon)]n^2(n+t)^2 - f(n,t,\varepsilon)f(n+1,t,\varepsilon)}{f(n,t,\varepsilon)f(n+1,t,\varepsilon)n^2(n+t)^2}
\]
\[= \frac{[9n^2 + (2a+9)n + (a+b+3)n^2(n+t)^2]}{f(n,t,\varepsilon)f(n+1,t,\varepsilon)n^2(n+t)^2}
\]
\[= - \frac{[3n^3 + (a+9)n^2 + (2a+b+9)n + (a+b+c+3+\varepsilon)]}{f(n,t,\varepsilon)f(n+1,t,\varepsilon)n^2(n+t)^2}.
\]

deleting

where \( A(t) \) is a function with variable \( t \), then we fix \( t \), for all \( \varepsilon > 0 \), there exists \( N_1 > 0 \), subject to

\[-6\varepsilon n^3 + A(t)O(n^2) = -6\varepsilon n^3 + O(n^2) < 0 \quad \text{for } n > N_1,
\]

hence

\[
\frac{1}{f(n,t,\varepsilon)} - \frac{1}{f(n+1,t,\varepsilon)} - \frac{1}{n^2(n+t)^2} < 0,
\]

and this completes the proof.

\( \square \)

**Lemma 5:** Let \( g(n,t,\varepsilon) = 3n^3 + an^2 + bn + c - \varepsilon \), then for all \( \varepsilon > 0 \), there exists \( N_2 > 0 \), subject to

\[
\frac{1}{g(n,t,\varepsilon)} > \frac{1}{g(n+1,t,\varepsilon)} + \frac{1}{n^2(n+t)^2} \quad \text{for } n > N_2.
\]

**Proof:** The proof is similar with the Lemma 3.1, we can easily deduce the result.

**The proof of Theorem 2:** By Lemma 1, Lemma 4 and Lemma 5, we have for all \( \varepsilon > 0 \), there exists \( N_3 = \max\{N_1, N_2\} > 0 \), subject to

\[
\frac{1}{f(n,t,\varepsilon)} < \sum_{k=n}^{\infty} \frac{1}{n^2(n+t)^2} < \frac{1}{g(n,t,\varepsilon)} \quad \text{for } n > N_3,
\]

(3.5)

hence

\[
3n^3 + an^2 + bn + c - \varepsilon < \left( \sum_{k=n}^{\infty} \frac{1}{n^2(n+t)^2} \right)^{-1} < 3n^3 + an^2 + bn + c + \varepsilon,
\]

which is equivalent to

\[\left( \sum_{k=n}^{\infty} \frac{1}{n^2(n+t)^2} \right)^{-1} \sim 3n^3 + an^2 + bn + c.\]

\( \square \)

3.3. **Proof of Theorem 3**

**Proof of Theorem 3:** According to the method of proving Theorem 2, it is enough to prove that there exists polynomial

\[B(n) = b_0n^l + b_{l-1}n^{l-1} + \cdots + b_1n + b_0,
\]

subject to

\[
\frac{1}{B(n)} - \frac{1}{B(n+1)} - \frac{1}{n'(n+t)^r} = \frac{O(n^{l-1})}{B(n)B(n+1)n'(n+t)^r}.
\]

(3.6)
The left side
\[ \frac{B(n + 1) - B(n)}{B(n)B(n + 1)} - \frac{1}{n'(n + t)^i} \]
\[ = \frac{[B(n + 1) - B(n)]n'(n + t)^i - B(n)B(n + 1)}{B(n)B(n + 1)n'(n + t)^i} \]

Let
\[ C(n) = B(n + 1) - B(n), \]
\[ D(n) = n'(n + t)^i, \]
\[ E(n) = B(n), \]
\[ F(n) = B(n + 1). \]

Therefore, it is enough to prove
\[ C(n)D(n) - E(n)F(n) = O(n^{l-1}), \]

(3.7)
hence we have the necessary condition (1):
\[ \alpha (C(n)D(n)) = \alpha (E(n)F(n)), \]

the notation \( \alpha(f(x)) \) means the order of \( f(x) \), then we have
\[ l = r + s - 1. \]

(3.8)
We note the number of coefficients is \( r + s \), then we have
\[ C(n)D(n) = \sum_{i=1}^{r+s-1} \left( b_i \sum_{j=0}^{i-1} \binom{i}{j} d^n \right) \left( n' \sum_{p=0}^{s} \binom{s}{p} t^{i-p} n^p \right) \]
\[ = \sum_{i=0}^{2r+2s-2} \sum_{k_{1} \leq r+s-k_{2}} \left( \binom{s}{k_{2} - r} t^{i+s-k_{2}} \sum_{j=k_{1}}^{r+s-1} b(j) \binom{j}{k_{1}} \right) n^i \]
and
\[ E(n)F(n) = \sum_{i=0}^{r+s-1} b_i n^i \left( \sum_{q=0}^{a} \sum_{j=0}^{q} \binom{q}{j} n^j \right) \]
\[ = \sum_{i=0}^{r+s-1} b_i n^i \left( \sum_{q=0}^{a} \sum_{j=q}^{r+s-1} b(j) \binom{j}{q} n^q \right). \]
\[
\begin{align*}
&= \sum_{i=0}^{2r+2s-2} \left( \sum_{\substack{k_1+k_2+r+s-1 \leq i \leq 0 \leq k_3 \leq 4}} b_{k_3} \sum_{j=k_4}^{r+s-1} b_j \binom{s}{k_4} \right) n^i \\
&= \sum_{i=0}^{2r+2s-2} b_{k_3} \sum_{j=k_4}^{r+s-1} b_j \binom{s}{k_4} n^i.
\end{align*}
\]

We get the necessary condition (2): If \( r + s - 1 \leq i \leq 2r + 2s - 2 \), the coefficients of \( C(n)D(n) \) and \( E(n)F(n) \) is equal, which is equivalent to the system of equations as follow:

\[
\sum_{0 \leq s_1 \leq s, 2r+2s-3 \leq r \leq 3} \binom{s}{k_4} \sum_{j=k_1+1}^{r+s-1} b_j \binom{s}{k_1} = \sum_{0 \leq s_1 \leq s, 2r+2s-3 \leq r \leq 3} b_{k_3} \sum_{j=k_4}^{r+s-1} b_j \binom{s}{k_4} n^i.
\]

where \( i = 2r + 2s - 2, 2r + 2s - 3, \cdots, r + s, r + s - 1 \).

We rewrite the system of equations (3.9) as

\[
\left\{
\begin{array}{l}
(r + s - 1)b_{r+s-1} = b_{r+s-1}^2, \\
[(r + s - 2) - 2b_{r+s-1}]b_{r+s-2} = f_1(b_{r+s-1}), \\
[(r + s - 3) - 2b_{r+s-1}]b_{r+s-3} = f_2(b_{r+s-1}, b_{r+s-2}), \\
\hspace{2cm} \cdots \\
-2b_{r+s-1}b_0 = f_{r+s}(b_{r+s-1}, b_{r+s-2}, \cdots, b_1).
\end{array}
\right.
\]

Obviously the first equation of (3.10) has two solutions \( b_{r+s-1} = 0 \) and \( b_{r+s-1} = r + s - 1 \), combination with (3.8) \( l = r + s - 1 \), then we have

\[
b_{r+s-1} = r + s - 1.
\]

Substitute (3.11) into the second equation of (3.10), we have the coefficient in the left side \([r + s - 2) - 2b_{r+s-1}]\) is not equal to 0, and the right side \( f_1(b_{r+s-1}) \) is a constant, then the second equation of (3.10) has unique solution.

Repeat the above process, for every equation of (3.10), the coefficient in the left side is never equal to 0, and the right side is always a constant, which implies the system of equations has unique solution, donates it by \( (b_{2s-1}, b_{2s-2}, b_{2s-3}, \cdots, b_1, b_0) \) with

\[
\begin{align*}
b_{r+s-1} &= r + s - 1, \\
b_{r+s-2} &= \frac{(r + s - 1)^2}{2(r + s)} [(2st - (r + s)] \\
b_{r+s-3} &= \frac{(r + s - 1)^2}{12(r + s)^2} [6s(r^2s - r^2 + 2rs^2 - 2rs + s^3 - s^2 - 2s)t^2 \\
&\quad - 6s(r + 3r^2s - r^2 + 3rs^2 - 2rs - 2r + s^3 - s^2 - 2s)t \\
&\quad + (2r^4 + 8r^3s - r^3 + 12r^2s^2 - 3r^2s - 3r^2 + 8rs^3 - 3rs^2 - 6rs + 2s^4 - s^3 - 3s^2)]
\end{align*}
\]
where we define $\binom{j}{i} = 0$ for all $j \in \mathbb{N}^+$. It is obvious that the coefficients $b_i$ are determined by $r, s$ and $t$, and the solution is corresponded to a polynomial with $(r+s-1)$-order, donates it by $B(n, r, s, t)$, and we can easily prove $B(n, r, s, t)$ satisfies (3.6). And then we have

$$\left(\sum_{k=n}^{\infty} \frac{1}{n(n+t)^s}\right)^{-1} \sim B(n, r, s, t).$$

\begin{proof}
\end{proof}

4. Conclusions

IN this paper we discuss the reciprocal sums of generalized Riemann zeta function within the range of integers, and we can also consider other functions within the range of integers.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

References


