A Fixed Point Iterative Scheme Based on Green’s Function for Numerical Solutions of Singular BVPs

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Abstract

We suggest a novel iterative scheme for solutions of singular boundary value problems (SB-VPs) that is obtained by embedding Green’s function into the Picard-Mann Hybrid (PMH) iterative scheme. This new scheme we will call in the paper as PMH–Green’s iterative scheme and prove its convergence towards a sought solution of certain SBVPs. We impose possible mild conditions on the operator or on the parameters involved in our scheme to obtain our main outcome. After this, we prove that this new iterative scheme is weak $w^2$–stable. Eventually, using two different numerical examples of SBVPs, we show that our new approach suggests highly accurate numerical solutions as compared the corresponding Picard–Green’s and Mann-Green’s iterative schemes.

Keywords: Solution; iteration scheme; boundary value problem; Green’s function, Banach space.

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1 Introduction

In some sense, all the real world phenomena can be essentially modeled in the form of differential equations having certain boundary conditions [1]. This is one of the reasons that the study of differential equations is too much important. One of the difficulties which arise naturally in studying differential equations is that their sought solutions are explicitly unknown (or it is very hard to solve them by using available analytical approaches). In this case, the sought is thus reasonably possible to set in the form of a fixed point of an operator (whose domain is possibly some distance space). However, when a fixed point of this operator exists then one naturally thinks that how to compute its approximate value using an appropriate numerical scheme. In 1922, Banach [2] proved
that if the domain of such operator is complete normed space and the operator is contraction then such operators admits a unique fixed point (which is the unique solution of the underlying problem) and the sequence of Picard [3] iterates essentially converges to this unique fixed point. This result has many useful applications in differential and integral equations because it give the existence and approximation of a solution for these problems. Notice that if $B$ denotes a Banach space with the norm $||.||$. Then the operator $F : B \rightarrow B$ is called a contraction (sometimes called a Banach-contraction) if for all $v, w \in B$ it is possible to find a real constant $\mu \in [0,1)$ such that
\[
||Fv - Fw|| \leq \mu||v - w||. \tag{1.1}
\]
We say that a point $s^* \in B$ is known as a fixed point of $F$ in the case when the equation $Fs^* = s^*$ holds and we shall write $fix(F)$ to denote set of all fixed points. In this case the Picard iteration [3] of $F$ is defined as:
\[
\begin{cases}
    w_0 \in B, \\
    w_{m+1} = Fw_m, (m = 0, 1, 2, 3, ...).
\end{cases} \tag{1.2}
\]

We know that the selfmap $F$ is known as a nonexpansive mapping on $B$ if the relation (1.1) holds for $\mu = 1$. Fixed point approximation under different iterative schemes is an active and important area of research on its own [4–7] and others. Although, Browder [8] (cf. also Gohde [9] and Kirk [10]) fixed point theorem suggests a fixed point (may not unique) for a certain nonexpansive operator in a Banach space setting. However, there are some well-known numerical examples of nonexpansive operators, for which the Picard iteration is not convergent to its fixed point (see, e.g., [11] and others). To overcome the case of nonexpansive mappings, Mann [12] suggested a new iteration scheme which need an initial value as well as a sequence of real numbers whose values are between 0 and 1.

The Mann iteration [12] recursively generates a sequence as:
\[
\begin{cases}
    w_0 \in B, \\
    w_{m+1} = (1 - \alpha_m)w_m + \alpha_mFw_m, (m = 0, 1, 2, 3, ...),
\end{cases} \tag{1.3}
\]
where $\alpha_m \in [0, 1]$.

In the literature of iterative schemes, it is known that the speed of the both Picard [3] and Mann [12] iteration is slow. To achieve a better rate of convergence, Khan [13] combined the iterative schemes due to Picard and Mann and named the resultant iterative scheme as a Picard–Mann hybrid iterative scheme (PMH–iterative scheme).

Precisely, PMH–iterative scheme [13] recursively generates a sequence as:
\[
\begin{cases}
    w_0 \in B, \\
    v_m = (1 - \alpha_m)w_m + \alpha_mFw_m, \\
    w_{m+1} = Fv_m, (m = 0, 1, 2, 3, ...),
\end{cases} \tag{1.4}
\]
where $\alpha_m \in (0, 1)$.

Khan proved that the PMH–iteration scheme (1.4) essentially converges to a fixed point of a certain operator. Moreover, he proved analytically and numerically that this scheme suggests high
accurate results corresponding to Mann and Picard iterations. Existence and approximation of solutions for BVPs is an important area on its own. Different techniques have been studied by authors for existence and approximation of solutions for various classes of BVPs [14, 15]. On the other hand, Khuri and Sayfy [16–18] embedded Green’s function into some well-known iterative schemes and proved that these new type of scheme suggests high accurate results corresponding to the other available methods of the literature. Motivated by Khuri and Sayfy, Assadi et al. [19] introduced Picard-Green’s and Mann-Green’s iterative schemes for a class of SBVPs and proved that both of these schemes are better than the many other previous iterative schemes studied for SBVPs. It is now a challenged question that whenever it possible to obtain a modified version of this scheme based on Green’s function for finding solution of SBVP? In this paper, we first obtain the requested version of this scheme for SBVPs and named it PMH–Green’s iterative scheme and show the PMH-Green produce very high accurate results compared to Mann-Green’s and Picard-Green’s iteration schemes. We also show that the PMH–Green’s iterative scheme is weak \( w^2 \)-stable in this case. The numerical computations given in the end of paper supports the main outcome of the paper and suggests the numerical effectiveness of the proposed scheme.

2 Overview of the iterative scheme

To propose the desired PMH-Green’s iterative scheme, first we some elementary concepts and results that which are necessary for the main work.

2.1 Construction of the Green’s function

This subsection will establish the Green’s function for a broad class of SBVPs. To succeed in this aim, suppose \( t \in (a, b) \), we consider a linear second order equation which is mathematically written as following:

\[
L(w) = w''(t) + p(t)w' + q(t)w(t) = f(t),
\]

and the associated boundary conditions (BCs) are the following:

\[
\begin{align*}
B_a[w] &= \alpha_0 w(a) + \alpha_1 w'(a) = \xi, \\
B_b[w] &= \beta_0 w(b) + \beta_1 w'(b) = \lambda.
\end{align*}
\]

It should be noted that the possible general solution is given as \( w(t) = w_h(t) + w_p(t) \), here the function \( w_h(t) \) is essentially the solution for the equation \( L[w] = 0 \) subjected to the BCs suggested in (2.2), and \( w_p(t) \) is solution for the equation \( L[w] = f(w) \) endowed with the homogeneous BCs given below:

\[
B_a[w] = B_b[w] = 0.
\]

Now for finding \( w_p(t) \), one needs a solution for

\[
L[w] = \delta(t - s),
\]
which is essentially subject to the BCs as given in (2.3) and in this case, such a solution is known as a Green’s function denoted normally by $G(t, s)$. Then

$$w_p = \int_a^b G(t, s)f(s)ds. \quad (2.5)$$

Let $w_1, w_2$ be two linearly independent solutions of $L[w] = 0$. Notice that the Green’s function essentially obeys the homogeneous equation for each choice of $t \neq s$ and thus it will be a linear combination of $w_1$ and $w_2$:

$$G(t, s) = \begin{cases} 
  c_1 w_1(t) + c_2 w_2(t) & \text{when } a < t < s \\
  d_1 w_1(t) + d_2 w_2(t) & \text{when } s < t < b,
\end{cases}$$

the constants $c_i$ and $d_i$, $(i = 1, 2)$ are determined using the following axioms:

$(A_1.)$ \quad $G$ satisfies the given homogeneous BCs, i.e.,

$$B_a[G(t, s)] = B_b[G(t, s)] = 0. \quad (2.6)$$

$(A_2.)$ \quad Continuity of $G$ at $t = s$:

$$c_1 w_1(s) + c_2 w_2(s) = d_1 w_1(s) + d_2 w_2(s). \quad (2.7)$$

$(A_3.)$ \quad Jump discontinuity of $G'$ at $t = s$, i.e.,

$$d_1 w'_1(s) + d_2 w'_2(s) - c_1 w'_1(s) - c_2 w'_2(s) = 1. \quad (2.8)$$

For nonlinear SBVPs

$$w''(t) + q(t)w'(t) + q(t)w(t) = f(t, w(t), w'(t)), \quad (2.9)$$

the particular solution satisfies

$$w_p = \int_a^b G(t, s)f(s, w_p(s), w'_p(s))ds, \quad (2.10)$$

and here $G$ is the Green’s function connected to (2.9).

### 2.2 PMH–Green’s iterative scheme

Now we propose our desired PMH–Green’s iterative scheme for approximate solutions of the following SBVPs of the form as follows:

$$L[w] = w'' + \frac{q}{t}w' = f(t, w, w'), \quad (2.11)$$

and the associated BCs are as given in (2.2). Now consider the Green’s function $G$ corresponding to the linear term and we consider the following operator:
A Fixed Point Iterative Scheme Based on Green’s Function

\[ M[w_p] = \int_a^b G(t, s)L[w_p]ds. \]  
(2.12)

Now from (2.10) and (2.12), we obtain

\[ M[w_p] = \int_a^b G(t, s)[L(w_p) - f(s, w(s), w'(s))]ds + w_p. \]  
(2.13)

Put \( w_p = w \) in (2.13), one has

\[ M(w) = w + \int_a^b G(t, s)[L(w) - f(s, w(s), w'(s))]ds. \]  
(2.14)

Hence from (2.14), we obtain the modified form of PMH-iterative scheme given in (1.4) as follows:

\[
\begin{align*}
    v_m &= (1 - \alpha_m)w_m + \alpha_m M[w_m], \\
    w_{m+1} &= M[v_m],
\end{align*}
\]  
(2.15)

which yields the following iterative procedure:

\[
\begin{align*}
    v_m &= (1 - \alpha_m)w_m + \alpha_m \int_a^b G(t, s)[L(w_m) - f(s, w_m(s), w'_m(s))]ds + w_m, \\
    w_{m+1} &= \int_a^b G(t, s)[L(v_m) - f(s, v_m(s), v'_m(s))]ds + v_m, (m = 0, 1, 2, 3, ...),
\end{align*}
\]  
(2.16)

It follows that

\[
\begin{align*}
    v_m &= w_m + \alpha_m \int_a^b G(t, s)[L(w_m) - f(s, w_m(s), w'_m(s))]ds, \\
    w_{m+1} &= v_m + \int_a^b G(t, s)[L(v_m) - f(s, v_m(s), v'_m(s))]ds, (m = 0, 1, 2, 3, ...),
\end{align*}
\]  
(2.17)

and here \( L \) denotes the linear term and the initial value to start the scheme, that is, \( w_0 \) must be chosen in a way that satisfies the equation (2.11), \( L[w] = 0 \), and as well as the given specified BCs.

3 Convergence result

We are now interested in establishing the main convergence result. For this, let \( q \geq 2 \) and consider a SBVP as provided below:

\[ w''(t) + \frac{q}{t}w'(t) = f(t, w(t), w'(t)), \]  
(3.1)

and the associated BCs are the following:

\[ w'(0) = \alpha, \quad w(1) = \beta. \]  
(3.2)

To construct the required Green’s function associated to (3.1), we will apply the axioms of Green’s function which we given in last section. Hence after solving the equation (3.1), one has

\[ G(t, s) = \begin{cases} 
    A + Bw^{1-q}, & \text{when } 0 < t < s \\
    C + Dw^{1-q}, & \text{when } s < t < 1.
\end{cases} \]
Accordingly, using homogenous BCs as given in (3.2), that is, \( w'(0) = w(1) = 0 \), one have the following

\[
B = 0, \quad D + C = 0. \tag{3.3}
\]

Applying the continuity axioms of Green’s function, we obtain

\[
A + Bs^{1-q} = C + Ds^{1-q}. \tag{3.4}
\]

Using unit jump discontinuity connected to the first derivative of the Green’s function, we have

\[
D(1 - q)s^{-q} - B(1 - q)s^{-q} = 1. \tag{3.5}
\]

After solving (3.3)–(3.5), we get the desired Green’s function as follows:

\[
G(t, s) = \begin{cases} \frac{s^\alpha - q}{q-1}, & \text{when } 0 < t < s \\ \frac{s^\alpha}{q-1}, & \text{when } s < t < 1. \end{cases}
\]

Now embedding the above Green’s function in the PMH–iterative scheme given in (2.17), we get the following PMH–iterative scheme:

\[
\begin{align*}
    v_m &= w_m + \alpha_m \int_0^t \frac{s^\alpha}{q-1} (t^1 - 1) [w_m''(s) + \frac{q}{s} w_m'(s) - f(s, w_m(s), w_m'(s))] ds,
    \\
    + \int_0^t \frac{1}{q-1} (s - s^\alpha) [w_m''(s) + \frac{q}{s} w_m'(s) - f(s, w_m(s), w_m'(s))] ds,
    \\
    w_{m+1} &= v_m + \int_0^t \frac{s^\alpha}{q-1} (t^1 - 1) [v_m''(s) + \frac{q}{s} v_m'(s) - f(s, v_m(s), v_m'(s))] ds,
    \\
    + \int_0^t \frac{1}{q-1} (s - s^\alpha) [v_m''(s) + \frac{q}{s} v_m'(s) - f(s, v_m(s), v_m'(s))] ds,
    \\
    (m = 0, 1, 2, 3, ...).
\end{align*}
\tag{3.6}
\]

It follows that

\[
\begin{align*}
    v_m &= w_m + \alpha_m \int_0^t G(t, s) [w_m''(s) + \frac{q}{s} w_m'(s) - f(s, w_m(s), w_m'(s))] ds,
    \\
    w_{m+1} &= v_m + \int_0^t G(t, s) [v_m''(s) + \frac{q}{s} v_m'(s) - f(s, v_m(s), v_m'(s))] ds,
    \\
    (m = 0, 1, 2, 3, ...).
\end{align*}
\tag{3.7}
\]

Set \( F_G : C[0, 1] \to C[0, 1] \) by

\[
F_Gw(t) = w(t) + \int_0^t G(t, s) [w_m''(s) + \frac{q}{s} w_m'(s) - f(s, w(s), w'(s))] ds.
\tag{3.8}
\]

Then (3.7) becomes

\[
\begin{align*}
    v_m &= (1 - \alpha_m)w_m + \alpha_m F_Gw_m,
    \\
    w_{m+1} &= F_Gv_m, (m = 0, 1, 2, 3, ...).
\end{align*}
\tag{3.9}
\]

The main result of the paper is now ready to establish.

**Theorem 3.1.** Consider a Banach space \( B = C[0, 1] \) with the supremum norm. Let \( F_G : B \to B \) be the operator defined in (3.8) and \( \{w_m\} \) be the sequence of PMH–Green’s iterative scheme (3.9). Assume that the following conditions hold:

\[
\text{Condition 1: } F_G \text{ is a contraction on } C[0, 1].
\]

\[
\text{Condition 2: } \lim_{m \to \infty} w_m(s) = w(s) \text{ uniformly on } [0, 1].
\]

Then the sequence \( \{w_m\} \) converges to the unique solution of the integral equation (3.3).
(a) \( \frac{1}{2(q-1)} M_c < 1 \), where \( M_c = \max \{0,1 \} \times \mathbb{R}^2 | \frac{\partial f}{\partial w} | \).

(b) \( \sum \alpha_m = \infty \) or for some \( \alpha \), \( 0 < \alpha \leq \alpha_m \).

Subsequently, \( \{w_m\} \) converges strongly to the unique solution of the problem (3.1)–(3.2).

**Proof.** By using assumption (a), we show that \( F_G \) is a Banach-contraction, that is, \( ||F_G v - F_G w|| \leq \mu ||v - w|| \) for all \( v, w \in B \) and some fixed \( \mu \in [0,1) \). To do this, direct integration gives us,

\[
\int_0^t \frac{s^q}{1-q} (t^{1-q} - 1) |w''(s) + \frac{q}{s} w'(s)| ds = \frac{(t^{1-q} - 1)}{1-q} \int_0^t [s^q w''(s) + qs^{q-1} w'(s)] ds
\]

\[= \frac{(t^{1-q} - 1)}{1-q} \int_0^t [(s^q w'(s)')] ds
\]

\[= \frac{t - t^q}{1-q} w'(t).
\] (3.10)

Also, integration by parts twice, we get

\[
\int_t^1 \frac{1}{q-1} (s^q - s) w''(s) = \frac{1}{q-1} [(1-q)w(1) + (qt^q - 1)w(t) + (t - t^q)w'(t)] + q(q-1) \int_t^1 s^{q-2} w(s) ds.
\] (3.11)

Now integrating once, we have

\[
\int_t^1 \frac{1}{q-1} (s^q - s) \frac{q}{s} w(s) ds = \frac{q}{q-1} [(1 - t^{q-1})w(t) - (q - 1) \int_t^1 s^{q-2} w(s) ds].
\] (3.12)

Put (3.10), (3.11) and (3.13), we have

\[
F_G(w) = w(1) + \int_0^t \frac{s^q}{1-q} (t^{1-q} - 1) f(s, w(s), w'(s)) ds + \int_t^1 \frac{1}{q-1} (s^q - s) f(s, w(s), w') ds.
\] (3.13)

From (3.2), \( w(1) = \beta \), therefore (3.13) takes the following form

\[
F_G(w) = \beta + \int_0^1 G(t, s) f(s, w(s), w'(s)) ds.
\] (3.14)

Hence

\[
|F_G(v) - F_G(w)| = \left| \beta + \int_0^1 G(t, s) f(s, v, v') ds - \beta - \int_0^1 G(t, s) f(s, w, w') ds \right|
\]

\[
= \int_0^1 G(t, s) |f(s, v, v') - f(s, w, w')| ds.
\] (3.15)

Now a simple integration suggests

\[
\int_0^1 G(t, s) ds = \frac{1}{2(q+1)} (t^2 - 1) = g(t).
\] (3.16)

It is easy to see that the \( g(t) \) attains the maximum value in \( [0,1] \) either at endpoints or on the critical points. Hence

\[
|g(t)| \leq \frac{1}{2(q+1)}.
\] (3.17)
By (3.15), (3.16) and (3.17), one has
\[ |F_G(v) - F_G(w)| \leq \frac{1}{2(q+1)} \max_{t \in [0,1]} |f(t, v, v') - f(t, w, w')| \]
\[ \leq \frac{1}{2(q+1)} M_c ||u - v||. \]  
(3.18)

But \( \mu = \frac{1}{2(q+1)} M_c < 1 \), it follows from (3.18), that \( F_G \) is a Banach contraction. Since \( B \) is complete and \( F_G \) is a Banach-contraction, thanks to the BCP [2], \( F_G \) admits essentially a unique fixed point in \( B = C[0,1] \) and this point we shall denote by \( s^* \) and hence it follows that this \( s^* \) is a unique solution of the problem (3.1)–(3.2).

Moreover, from the assumption (b), we will prove that PMH–Green’s iterative converges strongly to \( s^* \). First we assume the case when \( \sum \alpha_m = \infty \). Now
\[ ||v_m - s^*|| = ||(1 - \alpha_m)w_m + \alpha_m F_G w_m - s^*|| \]
\[ = ||(1 - \alpha_m)(w_m - s^*) + \alpha_m(F_G w_m - s^*)|| \]
\[ \leq (1 - \alpha_m)||w_m - s^*|| + \alpha_m||F_G w_m - s^*|| \]
\[ \leq (1 - \alpha_m)||w_m - s^*|| + \alpha_m \mu ||w_m - s^*|| \]
\[ = [1 - \alpha_m(1 - \mu)]||w_m - s^*||. \]

Hence we obtain
\[ ||v_m - s^*|| \leq [1 - \alpha_m(1 - \mu)]||w_m - s^*||. \]  
(3.19)

Finally, we use (3.19), we compute \( ||w_{m+1} - s^*|| \) as follows.
\[ ||w_{m+1} - s^*|| = ||F_G v_m - s^*|| \]
\[ \leq \mu ||v_m - s^*|| \]
\[ \leq \mu [1 - \alpha_m(1 - \mu)]||v_m - s^*||. \]

Hence, we obtain
\[ ||v_m - s^*|| \leq \mu^2 [1 - \alpha_m(1 - \mu)]||v_m - s^*||. \]  
(3.20)

Now from (3.20), we step by step obtain the following
\[ ||w_{m+1} - s^*|| \leq \mu [1 - \alpha_m(1 - \mu)]||w_m - s^*|| \]
\[ \leq \mu^2 [1 - \alpha_m(1 - \mu)][1 - \alpha_{m-1}(1 - \mu)]||w_{m-1} - s^*|| \]
\[ \leq \mu^3 [1 - \alpha_m(1 - \mu)][1 - \alpha_{m-1}(1 - \mu)][1 - \alpha_{m-2}(1 - \mu)]||w_{m-2} - s^*||. \]

It follows that
\[ ||w_{m+1} - s^*|| \leq (\mu)^{m+1} \Pi_{i=0}^{m}[1 - \alpha_i(1 - \mu)]||w_0 - w^*||. \]  
(3.21)

Noting that \( \lim_{m \to \infty} (\mu)^m = 0 \) because \( \mu \in [0,1) \). Also, it is well-known from the classical analysis that \( 1 - w \leq e^{-w} \) for all \( w \in [0,1] \). Taking these facts into account with (3.21), we get
\[ ||w_{m+1} - s^*|| \leq (\mu)^{m+1} e^{-(1-\mu) \sum_{i=0}^{m} \alpha_i}||w_0 - s^*||. \]  
(3.22)
As supposed $\sum \alpha_m = \infty$ and $\mu$ lies in $[0, 1)$, we have from (3.22) that

$$\lim_{m \rightarrow \infty} ||w_{m+1} - s^*|| = 0.$$ 

Accordingly, $\{w_m\}$ converges to a fixed point $s^*$ of $F_G$ which is the unique solution of the problem (3.1)–(3.2). The case when $0 < \alpha \leq \alpha_m$ is elementary and hence omitted.

4 Stability

In all branches of mathematics where iterative methods are used for finding approximate value of the sought solution, stability analysis is one of the desirable properties for such schemes (see., [20–22] and others). Fixed point procedure may or may not stable when we implement them on a certain operator equation [23] (cf. also [24, 25] and others). Suppose a given iterative scheme of a certain operator which is convergent to some of its fixed point. In this case, the iterative scheme is said to be stable if and only if the estimated error between two successive iterative strategy do not affects its so-called convergence. As many knows, stability for fixed points iterations finds its initial roots in the paper due to Urabe [26]. Motivated by Urabe [26], Harder and Hicks [27] constructed mathematical definition for stability. Some basic concepts which we need in the work are recalled below.

Definition 4.1. [27] Consider a mapping $F$ of a Banach space $B$ and suppose $\{w_m\} \subseteq B$ is a sequence generated from certain iterative scheme using the mapping $F$ as follows:

$$\begin{cases} 
  w_0 \in B, \\
  w_{m+1} = \gamma(F, w_m), 
\end{cases}$$

(4.1)

here, the element $w_0$ denotes starting point and $\gamma$ is a function of $F$ and $w_m$. Assume that the sequence of iterates $\{w_m\}$ converges to $s^* \in \text{fix}(F)$. In this case, $\{w_m\}$ is said to be stable if and only if

$$\lim_{m \rightarrow +\infty} ||s_{m+1} - \gamma(F, s_m)|| = 0 \Rightarrow \lim_{m \rightarrow +\infty} s_m = s^*,$$

where $\{s_m\}$ is any chosen sequence in the space $B$.

Definition 4.2. [28] Suppose $\{s_m\}$ and $\{w_m\}$ are any two sequences in a Banach space. We say that these two sequences equivalent if and only if $\lim_{m \rightarrow +\infty} ||s_m - w_m|| = 0$.

Opposed to the concept of arbitrary sequences, Timis [29], used the concept of equivalent sequences and obtained a new mathematical definition of the weak stability. This new type of stability is called the weak $w^2$-stability. The formal definition is given below.

Definition 4.3. [29] Consider a Banach space $B$ and $F$ a selfmap on $B$. If $\{w_m\}$ is a sequence of iterates of $F$ produced by the formula (4.1). Assume that $\{w_m\}$ is convergent to a point $s^* \in \text{fix}(F)$. Then $\{w_m\}$ is said to be weak $w^2$-stable if for every equivalent sequence $\{s_m\} \subseteq B$ of $\{w_m\}$, one has the following

$$\lim_{m \rightarrow +\infty} ||s_{m+1} - f(F, s_m)|| = 0 \text{ implies } \lim_{m \rightarrow +\infty} s_m = s^*.$$
Using the above concepts, we now show that our PMH-Green’s iterative scheme (3.9) is weak \( w^2 \)-stable.

**Theorem 4.4.** Let \( B \), \( F_G \) and \( \{w_m\} \) be as given in the Theorem 3.1. The \( \{w_m\} \) is essentially weak \( w^2 \)-stable with respect to \( F_G \).

**Proof.** To complete the proof, we consider any equivalent sequence \( \{s_m\} \) of \( \{w_m\} \), that is \( \lim_{m \to \infty} ||s_m - w_m|| = 0 \). Put

\[
\epsilon_m = ||s_{m+1} - F_G r_m||,
\]

where \( r_m = (1 - \alpha_m) s_m + \alpha_m F_G s_m \).

Assumed that \( \lim_{m \to +\infty} \epsilon_m = 0 \). First we compute the estimate \( ||r_m - v_m|| \). For this,

\[
||r_m - v_m|| = ||[(1 - \alpha_m) s_m + \alpha_m F_G s_m] - [(1 - \alpha_m) w_m + \alpha_m T_G w_m]||
\]

\[
= ||[(1 - \alpha_m)(s_m - w_m) + \alpha_m(F_G s_m - F_G w_m)]||
\]

\[
\leq (1 - \alpha_m)||s_m - w_m|| + \alpha_m ||F_G s_m - F_G w_m||
\]

\[
\leq (1 - \alpha_m)||s_m - w_m|| + \alpha_m \mu s_m - w_m||
\]

\[
\leq [1 - \alpha_m (1 - \mu)] ||s_m - w_m||.
\]

Consequently, we find

\[
||r_m - v_m|| \leq [1 - \alpha_m (1 - \mu)] ||s_m - w_m||. \tag{4.2}
\]

Keeping (4.2) in mind, we can proceed as follows:

\[
||s_{m+1} - s^*|| \leq ||s_{m+1} - w_{m+1}|| + ||w_{m+1} - s^*||
\]

\[
\leq ||s_{m+1} - F_G r_m|| + ||F_G r_m - w_{m+1}|| + ||w_{m+1} - s^*||
\]

\[
\leq \epsilon_m + ||F_G r_m - w_{m+1}|| + ||w_{m+1} - s^*||
\]

\[
= \epsilon_m + ||F_G r_m - F_G v_m|| + ||w_{m+1} - s^*||
\]

\[
\leq \epsilon_m + \mu ||r_m - v_m|| + ||w_{m+1} - s^*||
\]

\[
\leq \epsilon_m + \mu [1 - \alpha_m (1 - \mu)] ||r_m - w_m|| + ||w_{m+1} - s^*||.
\]

Subsequently, we obtain

\[
||s_{m+1} - s^*|| \leq \epsilon_m + \mu [1 - \alpha_m (1 - \mu)] ||s_m - w_m|| + ||w_{m+1} - s^*||. \tag{4.3}
\]

By assumptions, \( \lim_{m \to +\infty} \epsilon_m = 0 \) and \( \lim_{m \to +\infty} ||s_m - w_m|| = 0 \) because \( \{s_m\} \) is an equivalent sequence for \( \{w_m\} \). Also \( \lim_{m \to +\infty} ||w_m - w^*|| = 0 \) due to the convergence of \( \{w_m\} \) towards \( s^* \). Accordingly, from (4.3), \( \lim_{m \to +\infty} ||s_m - s^*|| = 0 \). This means that \( \{w_m\} \) generated by PMH-Green’s iterative scheme (3.9) is weak \( w^2 \)-stable with respect to the mapping \( F_G \).

\[\square\]

5 Numerical Example and Computations

In this section, we consider several numerical examples to show the high accurate numerical results produced by our proposed method.
Example 5.1. First we consider the following SBVP which is known as represents the the equilibrium of isothermal gas sphere [30]:

\[ w''(t) + \frac{2}{t} w'(t) = -w^5(t), \quad (5.1) \]

subjected with the BCs:

\[ w'(0) = 0, \quad w(1) = \sqrt{\frac{3}{4}}, \quad (5.2) \]

where \( 0 < t < 1 \). The exact solution of (5.1)–(5.2) is \( w(t) = \sqrt{\frac{3}{3+t^2}} \).

Now using Example 5.1, the proposed scheme takes the following form:

\[
\begin{aligned}
v_m &= w_m - \alpha_m \int_0^1 s^2(1 - \frac{1}{s})[w''_m(s) + \frac{2}{s} w'_m(s) + w^5_m(s)] ds \\
-x_m &= -\alpha_m \int_1^t s(s - 1)[w''_m(s) + \frac{2}{s} w'_m(s) + w^5_m(s)] ds \\
x_{m+1} &= v_m - \int_0^1 s^2(1 - \frac{1}{s})[v''_m(s) + \frac{2}{s} v'_m(s) + v^5_m(s)] ds \\
&\quad - \int_t^1 s(s - 1)[v''_m(s) + \frac{2}{s} v'_m(s) + v^5_m(s)] ds \\
\end{aligned}
\]

\[ (5.3) \]

Now for \( \alpha_m = 0.99 \), the values generated by Picard-Green’s, Mann-Green’s and PMH-Green’s iterative schemes in Tables 1–3. Clearly the PMH-Green’s iterative moving faster to the solution. While The Green’s function involved in the scheme (5.3) is provided in the Figure 1. The graphical comparison of the absolute errors in this case is given in Figures 2.
<table>
<thead>
<tr>
<th>m</th>
<th>Picard – Green</th>
<th>Mann – Green</th>
<th>PMH – Green</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.866025</td>
<td>0.866025</td>
<td>0.866025</td>
</tr>
<tr>
<td>1</td>
<td>0.946403</td>
<td>0.945599</td>
<td>0.976422</td>
</tr>
<tr>
<td>2</td>
<td>0.976769</td>
<td>0.976114</td>
<td>0.994334</td>
</tr>
<tr>
<td>3</td>
<td>0.989216</td>
<td>0.988807</td>
<td>0.997594</td>
</tr>
<tr>
<td>4</td>
<td>0.994451</td>
<td>0.994222</td>
<td>0.998199</td>
</tr>
<tr>
<td>5</td>
<td>0.996676</td>
<td>0.996554</td>
<td>0.998311</td>
</tr>
<tr>
<td>6</td>
<td>0.997626</td>
<td>0.997563</td>
<td>0.998332</td>
</tr>
<tr>
<td>7</td>
<td>0.998032</td>
<td>0.998001</td>
<td>0.998336</td>
</tr>
<tr>
<td>8</td>
<td>0.998206</td>
<td>0.998191</td>
<td>0.998337</td>
</tr>
<tr>
<td>9</td>
<td>0.998281</td>
<td>0.998273</td>
<td>0.998337</td>
</tr>
<tr>
<td>10</td>
<td>0.998313</td>
<td>0.998309</td>
<td>0.998337</td>
</tr>
</tbody>
</table>

Table 1: Convergence of iterates towards the numerical solution for $t = 0.1$.

<table>
<thead>
<tr>
<th>m</th>
<th>Picard – Green</th>
<th>Mann – Green</th>
<th>PMH – Green</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.866025</td>
<td>0.866025</td>
<td>0.866025</td>
</tr>
<tr>
<td>1</td>
<td>0.926917</td>
<td>0.926308</td>
<td>0.866028</td>
</tr>
<tr>
<td>2</td>
<td>0.947268</td>
<td>0.946831</td>
<td>0.958322</td>
</tr>
<tr>
<td>3</td>
<td>0.955153</td>
<td>0.954895</td>
<td>0.960316</td>
</tr>
<tr>
<td>4</td>
<td>0.958392</td>
<td>0.958250</td>
<td>0.960684</td>
</tr>
<tr>
<td>5</td>
<td>0.959755</td>
<td>0.959681</td>
<td>0.960753</td>
</tr>
<tr>
<td>6</td>
<td>0.960335</td>
<td>0.960297</td>
<td>0.960765</td>
</tr>
<tr>
<td>7</td>
<td>0.960583</td>
<td>0.960564</td>
<td>0.960768</td>
</tr>
<tr>
<td>8</td>
<td>0.960689</td>
<td>0.960679</td>
<td>0.960768</td>
</tr>
<tr>
<td>9</td>
<td>0.960734</td>
<td>0.960730</td>
<td>0.960768</td>
</tr>
<tr>
<td>10</td>
<td>0.960754</td>
<td>0.960751</td>
<td>0.960768</td>
</tr>
</tbody>
</table>

Table 2: Convergence of iterates towards the numerical solution for $t = 0.5$. 
Table 3: Convergence of iterates towards the numerical solution for $t = 0.9$.

<table>
<thead>
<tr>
<th>m</th>
<th>Picard – Green</th>
<th>Mann – Green</th>
<th>PMH – Green</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.866025</td>
<td>0.866025</td>
<td>0.866025</td>
</tr>
<tr>
<td>1</td>
<td>0.881451</td>
<td>0.881297</td>
<td>0.885099</td>
</tr>
<tr>
<td>2</td>
<td>0.885140</td>
<td>0.885061</td>
<td>0.886964</td>
</tr>
<tr>
<td>3</td>
<td>0.886451</td>
<td>0.886409</td>
<td>0.887284</td>
</tr>
<tr>
<td>4</td>
<td>0.886976</td>
<td>0.886953</td>
<td>0.887343</td>
</tr>
<tr>
<td>5</td>
<td>0.887194</td>
<td>0.887182</td>
<td>0.887354</td>
</tr>
<tr>
<td>6</td>
<td>0.887287</td>
<td>0.887281</td>
<td>0.887356</td>
</tr>
<tr>
<td>7</td>
<td>0.887326</td>
<td>0.887323</td>
<td>0.887356</td>
</tr>
<tr>
<td>8</td>
<td>0.887343</td>
<td>0.887342</td>
<td>0.887356</td>
</tr>
<tr>
<td>9</td>
<td>0.887351</td>
<td>0.887350</td>
<td>0.887356</td>
</tr>
<tr>
<td>10</td>
<td>0.887354</td>
<td>0.887353</td>
<td>0.887356</td>
</tr>
</tbody>
</table>

Figure 2: Comparison of various iterative schemes based on Green’s function for Example 5.1.
We finish the section with following example.

**Example 5.2.** Now we consider a SBVP whose exact solution is not known explicitly as follows:

\[
\frac{d^2 w}{dt^2} + \frac{2}{t} w'(t) = -e^{-w(t)}, \quad (5.4)
\]

**subjected to the BCs:**

\[
w'(0) = 0, \quad 2w(1) + w'(1) = 0, \quad (5.5)
\]

where \(0 < t < 1\).

The initial iterate \(w_0(t) = 0\) corresponding to \(w'' = 0\) and given BCs.

Now using Example 5.2, the proposed scheme takes the following form:

\[
\begin{align*}
v_m &= w_m - \alpha_m \int_0^1 s^2 \left( \frac{1}{2} - \frac{1}{t} \right) [w''_m(s) + \frac{2}{s} w'_m(s) + e^{-w_m(s)}] ds \\
&\quad - \alpha_m \int_1^t s \left( \frac{s}{2} - 1 \right) [w''_m(s) + \frac{2}{s} w'_m(s) + e^{-w_m(s)}] ds, \\
w_{m+1} &= v_m - \int_0^1 s^2 \left( \frac{1}{2} - \frac{1}{t} \right) [v''_m(s) + \frac{2}{s} v'_m(s) + e^{-v_m(s)}] ds \\
&\quad - \int_t^1 s \left( \frac{s}{2} - 1 \right) [v''_m(s) + \frac{3}{s} v'_m(s) + e^{-v_m(s)}] ds.
\end{align*}
\]

(5.6)

The Green’s function involved in the scheme (5.6) is provided in the Figure 3. The absolute error in this case is given in the Table 4. Again we see that the PMH-Green’s iterative approach is more accurate than the Picard-Green’s and Mann-Green’s iterative approaches for the problem (5.4)–(5.5).

Figure 3: Plot of Green’s function involved in the scheme (5.6).
<table>
<thead>
<tr>
<th>Values of $t$</th>
<th>Picard – Green</th>
<th>Mann – Green</th>
<th>PMH – Green</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>$2.67164 \times 10^{-8}$</td>
<td>$1.40478 \times 10^{-8}$</td>
<td>$7.77156 \times 10^{-16}$</td>
</tr>
<tr>
<td>0.2</td>
<td>$2.62110 \times 10^{-8}$</td>
<td>$1.37820 \times 10^{-8}$</td>
<td>$8.32667 \times 10^{-16}$</td>
</tr>
<tr>
<td>0.3</td>
<td>$2.53779 \times 10^{-8}$</td>
<td>$1.33440 \times 10^{-8}$</td>
<td>$7.77156 \times 10^{-16}$</td>
</tr>
<tr>
<td>0.4</td>
<td>$2.42312 \times 10^{-8}$</td>
<td>$1.27410 \times 10^{-8}$</td>
<td>$7.21645 \times 10^{-16}$</td>
</tr>
<tr>
<td>0.5</td>
<td>$2.27905 \times 10^{-8}$</td>
<td>$1.19835 \times 10^{-8}$</td>
<td>$6.93889 \times 10^{-16}$</td>
</tr>
<tr>
<td>0.6</td>
<td>$2.10813 \times 10^{-8}$</td>
<td>$1.10848 \times 10^{-8}$</td>
<td>$6.66134 \times 10^{-16}$</td>
</tr>
<tr>
<td>0.7</td>
<td>$1.91348 \times 10^{-8}$</td>
<td>$1.00613 \times 10^{-8}$</td>
<td>$6.10623 \times 10^{-16}$</td>
</tr>
<tr>
<td>0.8</td>
<td>$1.69877 \times 10^{-8}$</td>
<td>$8.93231 \times 10^{-9}$</td>
<td>$5.27356 \times 10^{-16}$</td>
</tr>
<tr>
<td>0.9</td>
<td>$1.46826 \times 10^{-8}$</td>
<td>$7.72027 \times 10^{-9}$</td>
<td>$4.99600 \times 10^{-16}$</td>
</tr>
</tbody>
</table>

Table 4: Absolute error between different iterations.

Figure 4: Comparison of various iterative schemes based on Green’s function for Example 5.2

6 Conclusions

We modified the PHM-iterative scheme by embedding a Green’s function of a certain SBVP. The convergence to a sought solution to a given SBVP is proved under some possible mild conditions. We proved the proposed iterative scheme is weak $w^2$-stable. Some numerical experiments are performed and it has been shown the numerical accuracy of the PMH–Green’s is more accurate.
Corresponding to the Picard–Green’s and Mann-Green’s iterative schemes. Since the proposed scheme is stable and gives high accurate numerical solutions in the setting of SBVPs, we conclude that our results improve and extend many other results of the literature due to various authors. The next aim of the authors is to use the proposed scheme for other BVPs which arise in nano-fluid and mathematical physics. Eventually, we point out the following:

(i) If we prove the mapping \( F_G \) a Kannan contraction, that is, for all \( v, w \), there exists a constant \( \mu \in [0, \frac{1}{2}] \) such that \( \|F_Gv - F_Gw\| \leq \mu(\|v - F_Gv\| + \|w - F_Gw\|) \), then \( F_G \) also admits a unique fixed point. Thus our iterative scheme can be used to approximate the solutions. However, the Kannan mappings are sometimes not continuous \cite{31}, therefore, it is not the good option in our case. Because the space \( C[0, 1] \) contains only continuous functions.

(ii) Our iterative scheme uses only one sequence of scalars \( \{\alpha_m\} \) and the condition we imposed on it is very simple in our convergence theorem. Thus, our iterative scheme needs few parameters to start. Moreover, it gives high accurate results for small values of this sequence.

(iii) The iterative scheme is proved weak \( w^2 \)-stable. We know that every stable iterative scheme is weak \( w^2 \)-stable but the converse is not generally true. Thus our result of weak \( w^2 \)-stability contains the case when one proves a stability result for our iterative scheme in the classical sense.

(iv) Since the Banach space used in this paper contains only continuous functions. Thus it will be a challenge problem for us to replace this Banach space by a Banach space which also contains some discontinuous functions.

Declarations section

Competing interests. We have no Competing interest.

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Authors’ Contributions. Muhammad Arshad gave the idea as a supervisor. Junaid Ahmad wrote the initial draft. Reny George edited the final version and approved for submission.

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References


