A Comprehensive Study on Fuzzy and Crisp Graph Indices: Generalized Formulae, Proximity and Accuracy Analysis

Muhammad Umar Mirza\(^1\), Rukhshanda Anjum\(^2\), Maged Z. Youssef\(^3\) and Turki Alsuraiheed\(^4\).

\(^1\) Department of Mathematics and Statistics, University of Lahore, Lahore, Pakistan.
\(^2\) Department of Mathematics and Statistics, University of Lahore, Lahore, Pakistan.
\(^3\) Department of Mathematics and Statistics, Collage of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), P.O.Box 65892, Riyadh 11566, Saudi Arabia.
\(^4\) Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451 Saudi Arabia.

* Correspondence: Email: rukhshanda.anjum@math.uol.edu.pk

Abstract: This article presents a set of generalized \(\varphi\)-dependent formulae for fuzzy and crisp versions of various graph indices, including the first and second Zagreb indices, harmonic index and Randic index. These formulae are applied to the identity graph of the commutative ring \(\mathbb{Z}_\varphi\) and the resulting indices are calculated using MATLAB software for 20 prime numbers. The generated data is used for machine learning using Python and Jupyter notebook to investigate the relationship between fuzzy and crisp indices. The article also includes the relationship between fuzzy and crisp indices in the form of six-degree polynomials and an error analysis.

Keywords: Fuzzy Graph, Topological Indices, Fuzzy Indices, Polynomial Regression, Machine Learning, Identity Graph

Mathematics Subject Classification: 05C72

1. Introduction

The study of graph theory and its applications in various fields has gained significant attention in recent years. Graph indices, which are numerical measures associated with graphs, have been widely used to capture the structural characteristics of graphs and analyze their properties. Among various types of graph indices, the Zagreb index, harmonic index and Randic index are important measures that provide insights into the structural features of graphs. A lot of literature is being published on topological indices these days and yet this area has the potential to accommodate new researchers. New topological indices are being defined every day, so research in this field is never-ending. Topological
indices were originally developed in the field of chemistry to understand the representations of different chemical structures, but now they are not limited to just chemical graph theory. For example, in [3] lower bounds of eccentric distance sum index of connected graph and cacti are presented; in [4] the same index is calculated for the bridge graph; and in [5] the same index is calculated for the thorn graph into polynomial form. Studies on the topological indices of tree graphs are also available. In [6], a study is conducted on the first Zagreb index of the tree graph and the upper bounds of the first Zagreb index of the tree graph. In [7], the eccentric distance sum of a tree is calculated. The eccentric distance sum index and eccentric connectivity index of unicyclic graphs are calculated in [8, 9]. In [10–13] topological indices of the bipartite graph, composite graph, windmill graph and Sierpinski graphs are presented. Topological indices of some graph operations can also be calculated and a study is done on this in [14]. In recent years, fuzzy graph theory has emerged as a powerful tool to model and analyze uncertain information in graphs. Fuzzy graph theory extends classical graph theory by allowing edges and nodes to have degrees of membership in the range [0, 1], which reflects the degree of uncertainty or fuzziness in real-world graphs. Fuzzy graph indices, which are extensions of classical graph indices to fuzzy graphs, have been developed to capture the uncertainty in graphs and provide a more comprehensive characterization of their structural properties. The history of algebra goes back to antiquity. The oldest surviving example is an Egyptian papyrus from around 1500 BCE, which describes how to calculate volumes using chords and arc length. The first recorded use of the word algebra dates from around 1000 BC. The term "algebra" comes from the title of the book written by Abu Ja’far Muhammad Ibn Musa al-Khwarizmi called "Kitab al-jabr wa’l muqabala", which is a Persian version of his Arabic book given a Latin translation in several European languages. It was later translated into other European languages using different titles, such as Liber abaci by Thabit ibn Qurra in 1111, Algebra by Umar Khayyam in 1150 and Algebra by Fibonacci in 1202. Ring theory is a branch of Mathematics that deals with the study of rings. The first known published reference to ring theory occurred in 1391, when the Italian Mathematician Gerolamo Cardano proposed a method of reducing fractions. Despite this interesting development in Mathematics, ring theory as an independent topic hadn’t been given much attention until around 2000 AD. A commutative ring is a type of ring that has an addition defined. The first use of this definition was by Euler in 1766; however, he did not use his definition to define a ring but rather the inverse image and semidirect sum of a ring. The first way to define a commutative ring was given by Freiberger in 1964, who used it as an equivalent to a direct sum. Today we define a commutative ring as a ring in which multiplication is commutative, or a commutative ring is an integral domain if it has no zero divisors. In this article we worked on the commutative ring $\mathbb{Z}_\wp$. We constructed the identity graph of $\mathbb{Z}_\wp$. A graph is the best tool to understand any kind of relationship between the elements of a set. Algebraic graph theory enables us to better visualize the elements of groups or rings. New graphs are being developed by different algebraic structures, as in [15] the commuting graph of quaternion and dihedral groups, in [16] some non-commuting graphs and in [17] inverse graphs of some finite groups. In [18], some very useful groups and subgroups are presented as graphs with examples and a lot of useful results are presented, especially for the identity graph of groups. In [19] studies on conjugate graphs of groups presented. In [20], the subgroup graph of groups is studied. In [22–27] studies on the non-commuting graph of quasi-dihedral groups, dihedral groups and finite groups in general are conducted. In [28, 29] the topological indices of the subgroup graphs of the symmetric group and dihedral groups are calculated. In this research paper, we focus on studying the fuzzy and crisp cases of four well-known graph indices,
namely the first fuzzy Zagreb index, second fuzzy Zagreb index, harmonic index and Randic index for the identity graph of the ring $\mathbb{Z}_\wp$, where $\wp$ is a prime number. The identity graph of the ring $\mathbb{Z}_\wp$ is a specific type of graph that has $\wp$ nodes and represents the additive group structure of the integers modulo $\wp$. We aim to investigate the relationships between these fuzzy and crisp indices and the prime number $\wp$ and explore the potential of using machine learning techniques to model these relationships. To achieve these objectives, we first generalize the existing definitions of the first fuzzy Zagreb index, second fuzzy Zagreb index, harmonic index and Randic index to the identity graph of the ring $\mathbb{Z}_\wp$. We develop new mathematical formulations for these fuzzy and crisp indices based on fuzzy set theory and graph theory concepts. We analyze the properties of these indices and provide insights into their behavior in different scenarios.

Next, we propose a machine learning approach to find the polynomials that show the relationship between these fuzzy and crisp indices and the prime number $\wp$. We collected a large data set of identity graphs of the ring $\mathbb{Z}_\wp$ for various prime numbers and applied machine learning algorithms such as polynomial regression to generate $\wp$ polynomials that show the relationship between these fuzzy and crisp indices. To assess the accuracy of these polynomials, we compare the indices computed using these polynomials with the exact indices that we have generalized.

The results of our study are expected to contribute to the field of fuzzy graph theory by providing a deeper understanding of the behavior of fuzzy and crisp indices in the context of the identity graph of the ring $\mathbb{Z}_\wp$. Moreover our machine learning approach can potentially be applied to other graph indices and graph structures, providing a valuable tool for analyzing and predicting the properties of complex graphs in real-world applications.

In the following sections of this research paper we will present the formal definitions and mathematical formulations of the first Zagreb index, second Zagreb index, harmonic index and Randic index for the identity graph of the ring $\mathbb{Z}_\wp$. We will also describe our methodology for collecting and preprocessing the data, developing the machine learning model and evaluating its performance. Finally we will present and discuss the results and provide insights into the implications and potential applications of our findings.

2. Preliminaries

The identity graph of the ring $\mathbb{Z}_\wp$ as defined in [30] is denoted by $Id(\mathbb{Z}_\wp)$, which is defined as the graph with a vertex set equal to the set of units in $\mathbb{Z}_\wp$ and two different vertices $\sigma$ and $\varsigma$ are adjacent if $\sigma\varsigma = 1$. and every vertex of $Id(\mathbb{Z}_\wp)$ is adjacent to the multiplicative identity of $\mathbb{Z}_\wp$. $Id(\mathbb{Z}_7)$, $Id(\mathbb{Z}_{13})$ and the generalized identity graph of $\mathbb{Z}_\wp$ are given in Figure 1. Fuzzy graph of identity graph of $\mathbb{Z}_\wp$ $Id_{fuzz}(\mathbb{Z}_\wp)$ is defined as a triplet $(\nu, \mu, \text{\&})$, where $\nu$ is a set of vertices and $\mu$ is a set of edges, where each edge $(\sigma, \varsigma) \in \mu$ and each vertex $\sigma, \varsigma \in \nu$ is associated with a membership value $\nu(\sigma), \mu(\sigma, \varsigma) \in [0, 1]$, called vertex weight $\nu$ and edge weight $\mu$ Fuzzy degree of a vertex. The fuzzy degree of a vertex $\sigma$ in $Id_{fuzz}(\mathbb{Z}_\wp)$ is the sum of the membership values of all edges incident to that vertex. It measures the degree of connectedness of a vertex in the graph and is given by the formula $N(\sigma) = \sum_{\sigma \in \nu, (\sigma, \varsigma) \in \mu} \mu(\sigma, \varsigma)$.

The crisp degree of a vertex in a simple graph is simply the number of edges connected to that vertex. Here we will represent the crisp degree of vertex $\sigma$ as $\xi(\sigma)$. A topological index is a numerical value that characterizes the topological structure of a graph. It can be used to study the chemical or physical properties of molecules represented by graphs. The formulae for crisp and fuzzy topological indices
Figure 1. $Id_{crisp}(Z_7)$, $Id_{crisp}(Z_{13})$ and genralized identity graph of $Z_\varphi$

are given in Table 1.

<table>
<thead>
<tr>
<th>Index Name</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crisp first Zagreb Index [31]</td>
<td>$M_1(Id_{crisp}(Z_\varphi)) = \sum_{x \in V} (\xi(x))^2$</td>
</tr>
<tr>
<td>Crisp second Zagreb Index [32]</td>
<td>$M_2(Id_{crisp}(Z_\varphi)) = \sum_{(\xi(x)\xi(\sigma))}$</td>
</tr>
<tr>
<td>Crisp first harmonic Index [33–36]</td>
<td>$H(Id_{crisp}(Z_\varphi)) = \sum_{x \in E} (\xi(\sigma))^2$</td>
</tr>
<tr>
<td>Crisp first randic Index [37]</td>
<td>$R(Id_{crisp}(Z_\varphi)) = \sum_{x \in V} (\xi(\sigma))^2$</td>
</tr>
<tr>
<td>Fuzzy First Zagreb Index [38]</td>
<td>$M(Id_{fuc}(Z_\varphi)) = \sum_{x \in V} (\xi(\sigma))^2$</td>
</tr>
<tr>
<td>Fuzzy Second Zagreb Index [38]</td>
<td>$M^*(Id_{fuc}(Z_\varphi)) = \frac{1}{2} \sum_{x \in V} \xi(\sigma)\xi(\tau)\xi(\xi(\tau)+\xi(\xi(\sigma)))$</td>
</tr>
<tr>
<td>Fuzzy Harmonic Index [39]</td>
<td>$H(Id_{fuc}(Z_\varphi)) = \frac{1}{2} \sum_{x \in V} \xi(\tau)\xi(\xi(\tau))\xi(\xi(\xi(\tau)))$</td>
</tr>
<tr>
<td>Fuzzy Randic Index [39]</td>
<td>$R(Id_{fuc}(Z_\varphi)) = 1/2 \sum_{x \in V} \xi(\sigma)\xi(\xi(\sigma))\xi(\xi(\xi(\sigma))$</td>
</tr>
</tbody>
</table>

Table 1. Table of Fuzzy and crisp Graph Indices

3. Generalization of topological indices into $\varphi$-dependent polynomials and formulae

In this section we will present the generalized formulae and $\varphi$ dependent polynomials to calculate the topological indices of $Id_{crisp}(Z_\varphi)$ for higher values of $\varphi$. Using these formulae and polynomials, the process of calculating and computing topological indices becomes way easier and faster.

3.1. Generalization of crisp topological indices

For the sake of simplicity we relabeled $Id(z_\varphi)$ in Figure 1 as the graph given below in Figure 2.

**Theorem 3.1.** The crisp first Zagreb index $M_1(Id_{crisp}(Z_\varphi)) = \varphi^2 - 7$ where as $\varphi \geq 5$

**Proof.** In $Id_{crisp}(Z_\varphi)$, $\xi(\varphi - 1) = 1$, $\xi(1) = \varphi - 2$ and $\xi(\sigma) = 2 \\forall \sigma \in V \land \sigma \neq 1, \varphi - 1 \xi(\varphi - 1) = 1, \xi(1) = \varphi - 2$. As the total number of vertices in $Id_{crisp}(Z_\varphi) = \varphi - 1$, the number of vertices with
degree 2 is $\varphi - 3$.

\[
M_1(Id_{\text{crisp}}(Z_{\varphi})) = \sum_{x \in V} (\xi(x))^2
\]
\[
= [\xi(1)^2 + (\xi(\varphi - 1))^2 + (\xi(2))^2 + (\xi(3))^2 \ldots \ldots (\xi(\varphi - 2))^2]
\]
\[
= (\varphi - 2)^2 + (1)^2 + (2^2 + 2^2 + 2^2 \ldots \ldots + 2^2(\varphi - 3) \text{ times})
\]
\[
= (\varphi - 2)^2 + 1 + 4(\varphi - 3)
\]
\[
= \varphi^2 - 7
\]

\[\square\]

**Theorem 3.2.** The crisp second Zagreb index $M_2(Id_{\text{crisp}}(Z_{\varphi})) = 2\varphi^2 - 7\varphi + 4$ where as $\varphi \geq 5$

**Proof.**

\[
M_2(Id_{\text{crisp}}(Z_{\varphi})) = \sum_{xy \in E} \xi(x)\xi(y)
\]
\[
= \xi(1)\xi(\varphi - 1) + \xi(1)\xi(2) + \xi(1)\xi(3) + \ldots \xi(1)\xi(\varphi - 2)
\]
\[
+ \xi(2)\xi(3) + \xi(4)\xi(5) + \ldots \xi(\varphi - 3)\xi(\varphi - 2)
\]
\[
= (\varphi - 2) + 2(\varphi - 2) + 2(\varphi - 2) + \ldots + 2(\varphi - 2) \text{ (\varphi - 3) times}
\]
\[
+ 2.2 + 2.2 + 2.2 + \ldots + 2.2 \text{ (\frac{\varphi - 3}{2}) times}
\]
\[
= p - 2 + 2(\varphi - 3)(\varphi - 2) + 4(\frac{\varphi - 3}{2})
\]
Theorem 3.3. The crisp Randic index $R[\text{Id}_{\text{crisp}}(Z_{\wp})] = \frac{4}{\sqrt{\wp-2}} + \frac{\wp-3}{\sqrt{2(\wp-2)}} + \frac{\wp-3}{4}$ where as $\wp \geq 5$

Proof.

$$R[\text{Id}_{\text{crisp}}(Z_{\wp})] = \sum_{xy \in E(Id(Z_{\wp}))} [\xi(\sigma)\xi(\varsigma)]^{\frac{1}{2}}$$
$$= [\xi(1)\xi(\wp-1)]^{\frac{1}{2}} + [\xi(1)\xi(2)]^{\frac{1}{2}} + [\xi(1)\xi(3)]^{\frac{1}{2}} + \ldots [\xi(1)\xi(\wp-2)]^{\frac{1}{2}}$$
$$+ [\xi(2)\xi(3)]^{\frac{1}{2}} + [\xi(4)\xi(5)]^{\frac{1}{2}} \ldots [\xi(\wp-3)\xi(\wp-2)]^{\frac{1}{2}}$$
$$= [\wp-2]^{\frac{1}{2}} + [2(\wp-2)]^{\frac{1}{2}} + [2(\wp-2)]^{\frac{1}{2}} \ldots + [2(\wp-2)]^{\frac{1}{2}} \quad (\wp-3 \text{ times})$$
$$+ [2.2]^{\frac{1}{2}} + [2.2]^{\frac{1}{2}} \ldots + [2.2]^{\frac{1}{2}} \quad (\wp-3) \text{ times}$$
$$= [\wp-2]^{\frac{1}{2}} + [2(\wp-2)]^{\frac{1}{2}} (\wp-3) + [4]^{\frac{1}{2}} (\wp-3)$$
$$= \frac{1}{\sqrt{\wp-2}} + \frac{\wp-3}{\sqrt{2(\wp-2)}} + \frac{\wp-3}{4}$$

\[ \square \]

Theorem 3.4. The crisp harmonic index $H[\text{Id}_{\text{crisp}}(Z_{\wp})] = \frac{2}{\wp}(\wp-3) + \frac{\wp-3}{4} + \frac{2}{\wp-1}$ where as $\wp \geq 5$

Proof. As $H[\text{Id}_{\text{crisp}}(Z_{\wp})] = \sum_{\sigma \varsigma \in E} \frac{2}{\xi(\sigma)+\xi(\varsigma)}$. In the $\text{Id}_{\text{crisp}}(Z_{\wp})$ vertex, one has an edge with $\wp-2$ vertices. Among those vertices, only the vertex ($\wp-1$) has degree one; all other vertices have degree 2, so vertex one is connected to $\wp-3$ vertices with degree 2 and one vertex with degree one. vertex one has degree $\wp-2$. There are $\frac{\wp-3}{2}$ edges in which the degree of both vertices is 2. Hence

$$H(G) = \frac{2}{\wp-2} + 2(\wp-3) + \frac{2}{2+2} (\wp-3 \times 2 \times 2) + \frac{2}{\wp-1}$$
$$= \frac{2}{\wp}(\wp-3) + \frac{\wp-3}{4} + \frac{2}{\wp-1}$$

\[ \square \]
3.2. Comparative Analysis of Generalized Crisp Topological Indices

In the ensuing subsection, we delve into a comparative examination of the generalized topological indices. These indices have been methodically calculated in line with four distinct theorems, notably Theorem 3.1, Theorem 3.2, Theorem 3.3, and Theorem 3.4. To explain the nuanced variations and different applications of these generalized indices, Figure 3 and Table 2 is provided in this subsection. This visual depiction will aid in illustrating the relative strengths and unique qualities of these indices in diverse analytical scenarios.

Figure 3. Comparative Analysis of Generalized Crisp Topological Indices.

<table>
<thead>
<tr>
<th>Primes</th>
<th>Crisp Randic Index</th>
<th>Crisp Harmonic Index</th>
<th>First Crisp Zagreb Index</th>
<th>Second Crisp Zagreb Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>2.7121</td>
<td>2.4762</td>
<td>42.0000</td>
<td>53.0000</td>
</tr>
<tr>
<td>11</td>
<td>4.2190</td>
<td>3.6545</td>
<td>114.0000</td>
<td>169.0000</td>
</tr>
<tr>
<td>13</td>
<td>4.9335</td>
<td>4.2051</td>
<td>162.0000</td>
<td>251.0000</td>
</tr>
<tr>
<td>17</td>
<td>6.3142</td>
<td>5.2721</td>
<td>282.0000</td>
<td>463.0000</td>
</tr>
</tbody>
</table>

Table 2. Computed Crisp Topological Indices
3.3. Generalization of fuzzy topological indices

For the sake of simplicity, we relabel \(Id(z_\varphi)\) in figure 1 as the graph given below in figure 4 as a generalized fuzzy \(Id(z_\varphi)\).

![Diagram showing the relabeled graph](image)

**Figure 4.** Relabeled \(Id_{fuz}(Z_\varphi)\).

**Theorem 3.5.** The first fuzzy Zagreb index

\[
M[Id_{fuz}(Z_\varphi)] = \frac{(\varphi - 1) + (\varphi - 2)^2}{\varphi^3} + 1 \sum_{\varphi^3} (2i(2i + 1) + (2i + 1)^3).
\]

where as \(\varphi \geq 5\)

**Proof.**

\[
M[Id_{fuz}(Z_\varphi)] = \sum \varphi(\varphi^3)N(\varphi^3)^2
\]

\[
= \varphi(\varphi^3)N(\varphi^3)^2 + \varphi(\varphi^3)N(\varphi^3)^2 + \varphi(\varphi^3)N(\varphi^3)^2 + \varphi(\varphi^3)N(\varphi^3)^2 + \varphi(\varphi^3)N(\varphi^3)^2 + \varphi(\varphi^3)N(\varphi^3)^2 + \varphi(\varphi^3)N(\varphi^3)^2
\]

\[
= \frac{(\varphi - 1) + (\varphi - 2)^2}{\varphi^3} + \frac{1}{\varphi^3} \sum_{\varphi^3} (2i(2i + 1) + (2i + 1)^3).
\]
Theorem 3.6. The second fuzzy Zagreb index $M^*(Id_{fu}(Z_\varphi)) = \frac{1}{2\varphi^4} \left[ \sum_{\sigma \in E} \varpi(\sigma)N(\sigma)\varpi(\zeta)N(\zeta) \right]$

Proof.

\[
M^*(Id_{fu}(Z_\varphi)) = \frac{1}{2} \left[ \sum_{\sigma \in E} \varpi(\sigma)N(\sigma)\varpi(\zeta)N(\zeta) \right]
\]

\[
= \frac{1}{2} \left[ \varpi(1)N(1)\varpi(\varphi - 1)N(\varphi - 1) + \varpi(1)N(1)\varpi(2)N(2) + \varpi(1)N(1)\varpi(3)N(1) + ...
\]

\[
... + \varpi(1)N(1)\varpi(\varphi - 2)N(\varphi - 2) + \varpi(2)N(2)\varpi(3)N(3) + \varpi(4)N(4)\varpi(5)N(5) + ...
\]

\[
+ \varpi(\varphi - 3)N(\varphi - 3)\varpi(\varphi - 2)N(\varphi - 2)
\]

\[
= \frac{1}{2\varphi^4} \left[ (\varphi - 2)(\varphi - 1) + (\varphi - 2)2.3 + (\varphi - 2)3.3 + ... + (\varphi - 2)(\varphi - 2)(\varphi - 2) + 2.3.3.3 + 4.5.5.5 + ...
\]

\[
......(\varphi - 3)(\varphi - 2)(\varphi - 2)(\varphi - 2)
\]

\[
= \frac{1}{2\varphi^4} \left[ (\varphi - 2)((\varphi - 1) + 2.3 + 3.3 + 4.5 + 5.5 + ... + (\varphi - 3)(\varphi - 2) + (\varphi - 2)(\varphi - 2))
\]

\[
+ (2.3^3 + 4.5^3 + ... + (\varphi - 3)(\varphi - 2)^3)
\]

\[
= \frac{1}{2\varphi^4} \left[ (\varphi - 2)((\varphi - 1) + \sum_{i=1}^{\varphi-3} (2i(2i + 1) + (2i + 1)(2i + 1)) + \sum_{i=1}^{\varphi-3} 2i(2i + 1)^3 \right]
\]

\[\square\]

Theorem 3.7. The fuzzy harmonic index $H(Id_{fu}(Z_\varphi)) = \frac{\varphi^2}{2} \left[ \frac{1}{2\varphi} \sum_{i=1}^{\varphi-3} \left( \frac{1}{p-1+2i(2i+1)} + \frac{1}{p-1+(2i+1)2i} \right) \right] + \frac{4i+1}{2(p(2i+1)^2)} \right] \] where as $\varphi \geq 5$
Proof.

\[ H(Id_{fuc}(Z_\rho)) = \frac{1}{2} \left[ \sum_{\sigma, \varsigma \in \mathcal{E}} \left( \frac{1}{\sigma(\sigma) \mathcal{N}(\sigma) + \sigma(\varsigma) \mathcal{N}(\varsigma)} \right) \right] \]

\[ = \frac{1}{2} \left[ \left( \frac{1}{\sigma(1)d(1) + \sigma(\rho - 1)d(\rho - 1)} + \frac{1}{\sigma(1)d(1) + \sigma(2) + d(2)} \right) \right. \]

\[ + \frac{1}{\sigma(1)d(1) + \sigma(3) + d(3)} + \left. \cdots + \frac{1}{\sigma(1)d(1) + \sigma(\rho - 3) + d(\rho - 3)} \right) \]

\[ + \frac{1}{\sigma(1)d(1) + \sigma(\rho - 2) + d(\rho - 2)} + \frac{1}{\sigma(2)d(2) + \sigma(3) + d(3)} + \frac{1}{\sigma(4)d(4) + \sigma(5) + d(5)} \]

\[ + \ldots + \frac{1}{\sigma(\rho - 3)d(\rho - 3) + \sigma(\rho - 2) + d(\rho - 2)} \]

\[ = \frac{1}{2} \left[ \left( \frac{1}{\varphi - 1} + \frac{1}{\varphi} + \frac{1}{\varphi^{n-1}} + \frac{1}{\varphi^{n}} + \frac{1}{\varphi^{n+1}} + \frac{1}{\varphi^{n+2}} + \cdots \right) \right] \]

\[ = \frac{\varphi^2}{2} \left[ \left( \frac{1}{2\varphi - 1} \right) + \frac{1}{\varphi + 1} + \frac{1}{\varphi + 3} \right. \]

\[ + \frac{1}{\varphi + 5} + \frac{1}{\varphi + 7} + \left. \cdots + \frac{1}{(\varphi - 3)(\varphi - 2)(\varphi - 2)} \right] \]

\[ = \frac{p^2}{2} \left[ \frac{1}{2p - 1} + \sum_{i=1}^{\varphi^2-1} \left( \frac{1}{\varphi - 1 + 2i(2i + 1)} + \frac{1}{(\varphi - 1) + (2i + 1)^2} \right) \right. \]

\[ + \sum_{i=1}^{\varphi^2-1} \frac{1}{2i(2i + 1) + (2i + 1)^2} \]

\[ = \frac{p^2}{2} \left[ \frac{1}{2\varphi - 1} \sum_{i=1}^{\varphi^2-1} \left( \frac{1}{\varphi - 1 + 2i(2i + 1)} + \frac{1}{(\varphi - 1) + (2i + 1)^2} \right) \right. \]

\[ + \left. \sum_{i=1}^{\varphi^2-1} \frac{1}{2i(2i + 1) + (2i + 1)^2} \right] \]

\[ = \frac{p^2}{2} \left[ \frac{1}{2\varphi - 1} \sum_{i=1}^{\varphi^2-1} \left( \frac{1}{\varphi - 1 + 2i(2i + 1)} + \frac{1}{(\varphi - 1) + (2i + 1)^2} \right) \right. \]

\[ + \left. \sum_{i=1}^{\varphi^2-1} \frac{1}{2i(2i + 1) + (2i + 1)^2} \right] \]

\[ = \frac{\varphi^2}{2} \left[ \frac{1}{2\varphi - 1} \sum_{i=1}^{\varphi^2-1} \left( \frac{1}{\varphi - 1 + 2i(2i + 1)} + \frac{1}{\varphi - 1 + (2i + 1)^2} \right) + \frac{4i + 1}{2(i + 1)^2} \right] \]
Theorem 3.8. the second fuzzy Randic index

\[
R(Id_{\text{fuz}}(Z_{\varphi})) = \frac{\varphi - 2}{\varphi^4} \left( (\varphi - 1) \sum_{k=1}^{\varphi-2} (2k(2k+1) + (2k+1)^2) + \sum_{k=1}^{\varphi-2} 2k(2k+1)^3 \right)
\]
\[
+ \left( \sum_{k=1}^{\varphi-2} 2k(2k+1) \left( \sum_{\tau=4}^{2\tau+1} [2\tau(2\tau+1) + (2\tau+1)^2] + (\varphi - 1) \right) \right)
\]
\[
+ \left[ \sum_{k=1}^{\varphi-2} 2k(2k+1)^2 \left( \sum_{\tau=4}^{2\tau+1} (2\tau(2\tau+1) + (2\tau+1)^2) + (\varphi - 1) \right) \right]
\]

where as \( \varphi \geq 5 \)

Proof.

\[
R(Id_{\text{fuz}}(Z_{\varphi})) = \frac{1}{2} \left[ \sum_{\sigma, \varsigma \in V} \sigma(\sigma)N(\varsigma)\sigma(\varsigma)N(\varsigma) \right]^{1/2}
\]
\[
= \frac{1}{2} \left[ \sum_{\sigma \in E} \sigma(\sigma)N(\varsigma)\sigma(\sigma)N(\varsigma) + \sum_{\sigma \notin E} \sigma(\sigma)N(\sigma)\sigma(\varsigma)N(\varsigma) \right]^{1/2}
\]
\[
= \frac{1}{2} \left[ 2M^*(G) + \sum_{\sigma \notin E} \sigma(\sigma)N(\sigma)\sigma(\varsigma)N(\varsigma) \right]^{1/2}
\]
\[
R(Id_{\text{fuz}}(Z_{\varphi})) = \frac{1}{2} \left[ 2M^*(G) + \sum_{\sigma \notin E} \sigma(\sigma)N(\sigma)\sigma(\varsigma)N(\varsigma) \right]^{1/2} \tag{3.1}
\]

To calculate \( \sum_{\sigma \notin E} \sigma(\sigma)N(\sigma)\sigma(\varsigma)N(\varsigma) \) we can see that non adjacent pair of vertices are,

(2, 4), (2, 5), (2, 6), ..., (2, \( \varphi - 3 \)), (2, \( \varphi - 2 \)), (2, \( \varphi - 1 \)), (3, 4), (3, 5), (3, 6), ..., (3, \( \varphi - 3 \)), (3, \( \varphi - 2 \)), (3, \( \varphi - 3 \)),
\[
\sum_{\sigma \in \mathcal{E}} \sigma(\pi) N(\sigma) \sigma(\varsigma) N(\varsigma) = \frac{2}{\wp} \frac{3}{\wp} \frac{4}{\wp} \frac{5}{\wp} + \frac{2}{\wp} \frac{3}{\wp} \frac{5}{\wp} + \cdots + \frac{2}{\wp} \frac{3}{\wp} \frac{\wp - 3}{\wp} \frac{\wp - 2}{\wp} + \frac{2}{\wp} \frac{3}{\wp} \frac{\wp - 2}{\wp} + \frac{2}{\wp} \frac{\wp - 1}{\wp}
\]

\[
+ \frac{3}{\wp} \frac{3}{\wp} \frac{4}{\wp} \frac{5}{\wp} + \frac{3}{\wp} \frac{3}{\wp} \frac{5}{\wp} + \cdots + \frac{3}{\wp} \frac{3}{\wp} \frac{\wp - 3}{\wp} \frac{\wp - 2}{\wp} + \frac{3}{\wp} \frac{3}{\wp} \frac{\wp - 2}{\wp} + \frac{3}{\wp} \frac{\wp - 1}{\wp}
\]

\[
+ \frac{4}{\wp} \frac{5}{\wp} \frac{6}{\wp} \frac{7}{\wp} + \frac{4}{\wp} \frac{5}{\wp} \frac{7}{\wp} + \cdots + \frac{4}{\wp} \frac{5}{\wp} \frac{\wp - 3}{\wp} \frac{\wp - 2}{\wp} + \frac{4}{\wp} \frac{5}{\wp} \frac{\wp - 2}{\wp} + \frac{4}{\wp} \frac{\wp - 1}{\wp}
\]

\[
= \frac{1}{\wp^2} \left[ 2.3.4.5 + 2.3.5.5 \cdots (2.3)(\wp - 3)(\wp - 2)(2.3)(\wp - 2) + 2.3(\wp - 1) \right]
\]

3.3.4.5 + 3.3.5.5...(3 - 3)(\wp - 3)(\wp - 2) + (3.3)(\wp - 2)(\wp - 2) + 3.3(\wp - 1)

\[
+ 4.5.6.7 + 4.5.7.7 + \cdots + (4.5)(\wp - 3)(\wp - 2) + (4.5)(\wp - 2)(\wp - 2) + 4.5(\wp - 1)
\]

\[
+ 5.5.6.7 + 5.5.7.7 + \cdots + (5 - 5)(\wp - 3)(\wp - 2) + (5.5)(\wp - 2)(\wp - 2) + (5.5)(\wp - 1)
\]

\[
\text{added}
\]
\[
\sum_{\sigma \in E} \sigma(\sigma) \mathcal{N}(\sigma) \mathcal{F}(\sigma) = \frac{1}{\varphi^4} \left[ \sum_{k=1}^{\varphi^3} 2\kappa(2\kappa + 1)(\sum_{\tau=\kappa+1}^{\varphi^3} 2\tau(2\tau + 1) + (2\tau + 1)^2) + (\varphi - 1) \right. \\
+ \left. \left( \sum_{k=1}^{\varphi^3} 2\kappa(2\kappa + 1)^2 \sum_{\tau=\kappa+1}^{\varphi^3} (2\tau(2\tau + 1) + (2\tau + 1)^2) + (\varphi - 1) \right) \right].
\]

\[
R(\text{Id}_{fuz}(Z_{\varphi})) = \frac{\varphi - 2}{\varphi^4} \left[ (\varphi - 1) \sum_{k=1}^{\varphi^3} 2\kappa(2\kappa + 1) + (2\kappa + 1)^2 \right] + \sum_{k=1}^{\varphi^3} 2\kappa(2\kappa + 1)^3 \\
+ \left( \sum_{k=1}^{\varphi^3} 2\kappa(2\kappa + 1) \left( \sum_{\tau=\kappa+1}^{\varphi^3} 2\tau(2\tau + 1) + (2\tau + 1)^2 \right) + (\varphi - 1) \right]
\]

<table>
<thead>
<tr>
<th>Index</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fuzzy First Zagreb Index</td>
<td>[ M(\text{Id}<em>{fuz}(Z</em>{\varphi})) = \sum_{\sigma \in E} \sigma(\sigma) \mathcal{N}(\sigma)^2 ]</td>
</tr>
<tr>
<td>Fuzzy Second Zagreb Index</td>
<td>[ M^*(\text{Id}<em>{fuz}(Z</em>{\varphi})) = \frac{1}{2} \sum_{\sigma, \varsigma \in E} \sigma(\sigma) \mathcal{N}(\sigma) \mathcal{F}(\sigma) \mathcal{N}(\varsigma) ]</td>
</tr>
<tr>
<td>Fuzzy Harmonic Index</td>
<td>[ H(\text{Id}<em>{fuz}(Z</em>{\varphi})) = \frac{1}{2} \left( \sum_{\sigma, \varsigma \in E} \frac{1}{\sigma(\sigma) \mathcal{N}(\sigma) \mathcal{F}(\sigma) \mathcal{N}(\varsigma)} \right)^{-1/2} ]</td>
</tr>
<tr>
<td>Fuzzy Randic Index</td>
<td>[ R(\text{Id}<em>{fuz}(Z</em>{\varphi})) = \frac{1}{2} \left( \sum_{\sigma, \varsigma \in E} \sigma(\sigma) \mathcal{N}(\sigma) \mathcal{F}(\sigma) \mathcal{N}(\varsigma) \right)^{-1/2} ]</td>
</tr>
</tbody>
</table>

Table 3. Fuzzy Graph Indices

3.4. Comparative Analysis of Generalized Fuzzy Topological Indices

In the subsequent subsection, we dig into a comparative evaluation of the generalized topological fuzzy indices. These indices have been meticulously determined in conformity with four distinct theorems, especially Theorem 3.5, Theorem 3.6, Theorem 3.7, and Theorem 3.8. To clarify the nuanced variations and diverse applications of these generalized indices, Figure 5 and Table 4 is provided in this paragraph. This graphic depiction will aid in presenting the relative strengths and unique properties of various indices in diverse analytical contexts.
4. Use of machine learning to find relationships between fuzzy and crisp topological indices

We generated topological indices using mathematical formulas and polynomials, yielding numerical descriptions of the identity graph of $\mathbb{Z}_\wp$. These indices give light on the mathematical characteristics of the ring $\mathbb{Z}_\wp$. Our investigation extended to employing machine learning, specifically polynomial regression, to build a polynomial equation explaining the link between fuzzy and crisp topological indices. This technique describes this relationship as an nth-degree polynomial function.
4.1. Relationship Between fuzzy first Zagreb index and crisp first Zagreb index

The equation 4.1 shows the relationship between fuzzy first Zagreb index and crisp first Zagreb index. In this equation $x$ is the fuzzy first Zagreb index and $y$ is the crisp first Zagreb index.

\[
y = \frac{47939412149347 x^6}{73786976294838206464} - \frac{6857742982524237 x^5}{3093798587684389} + \frac{1475739525896412928}{5132010386538057} x^4 - \frac{5820187159405795 x^3}{288230376151711744} + \frac{569009884489911 x^2}{35184372088832} + \frac{140737488355328 x}{112589906842624}
\]  

(4.1)

The error analysis given in the figure shows the absolute error between the exact crisp first zagreb index and the approximated crisp first zagreb index by putting the values of fuzzy first zagreb index in equation 4.1.

Figure 6. Absolute error between exact crisp first zagreb index and approximated crisp first zagreb index by putting the values of fuzzy first zagreb index in equation 4.1.
4.2. Relationship Between Fuzzy 2nd Zagreb Index and Crispy 2nd Zagreb Index

The equation 4.2 shows the relationship between fuzzy 2nd Zagreb index and crisp 2nd Zagreb index. In this equation \( x \) is the fuzzy 2nd Zagreb index and \( y \) is the crisp 2nd Zagreb index.

\[
y = -\frac{4198636757307237 x^6}{36028797018963968} + \frac{3821466390892453 x^5}{2251799813685248} + \frac{2802698852264907 x^4}{281474976710656} + \frac{3821466390892453 x^3}{281474976710656} + \frac{2802698852264907 x^2}{3290590709507179} + \frac{1099082828796479 x}{4398046511104} + \frac{35184372088832}{5076600172600843} \] (4.2)

The error analysis given in the figure shows the absolute error between the exact crisp 2nd Zagreb index and the approximated 2nd Zagreb index by putting the values of the fuzzy 2nd Zagreb index in equation 4.2.

![Figure 7. Absolute error between exact crisp 2nd zagreb index and approximated crisp 2nd zagreb index by putting the values of fuzzy 2nd zagreb index in equation 4.2.](image-url)
4.3. Relationship Between Fuzzy Randic Index and Crispy Randic Index

The equation 4.3 shows the relationship between fuzzy Randic index and crisp 2nd Randic index. In this equation $y$ is the fuzzy Randic index and $x$ is the crisp Randic index.

$$y = \frac{3387822566640705 x^6}{4294967296} - \frac{613321410237291 x^5}{536870912} + \frac{5283103041825209 x^4}{8589934592} - \frac{34359738368 x^3}{137438953472} + \frac{3026309911703795 x^2}{3464515882695811} - \frac{3464515882695811 x}{2199023255552}$$

(4.3)

The error analysis given in the figure shows the absolute error between the exact crisp Randic index and the approximated crisp Randic index by putting the values of fuzzy Randic index in equation 4.3.

![Figure 8. Absolute error between exact crisp Randic index and approximated crisp Randic index by putting the values of fuzzy Randic index in equation 4.3.](image)
4.4. Relationship Between Fuzzy Harmonic Index and Crispy Harmonic Index

The equation 4.4 shows the relationship between fuzzy harmonic index and crisp harmonic index. In this equation $y$ is the fuzzy harmonic index and $x$ is the crisp harmonic index.

$$y = \frac{403444852988239}{536870912} x^6 - \frac{584110388243953}{536870912} x^5 + \frac{5028519656072005}{8589934592} x^4 - \frac{82120743106149}{536870912} x^3 + \frac{5742473945352849}{274877906944} x^2 - \frac{754343538853617}{274877906944} x + \frac{140737488355328}{140737488355328}$$

(4.4)

The error analysis given in the figure shows the absolute error between the exact crisp harmonic index and the approximated crisp harmonic index by putting the values of the fuzzy harmonic index in equation 4.4.

![Error vs Crisp Harmonic Index](image)

**Figure 9.** Absolute error between exact crisp harmonic index and approximated crisp harmonic index by putting the values of fuzzy harmonic index in equation 4.4.
5. Entropy Analysis of Fuzzy and Crisp Harmonic Indices

A histogram is created based on the values in Table 5 to visualize the distribution of values of crisp and fuzzy harmonic index, where each value have a frequency of 1. The histogram is normalized by calculating the probability \( p_\phi \) for each value \( \phi \), which is obtained by dividing 1 by the total number of data points (22 unique values). Using the entropy formula \( H = -\sum (p_\phi \times \log_2(p_\phi)) \) we can calculate entropy for one value and multiply it by the total number of unique values.

\[
\hat{H} \approx -\log_2(22) \approx 4.4594 \text{ bits}
\]

The calculated entropy for the crisp and fuzzy Harmonic index data set is approximately 4.4594 bits, indicating a high level of diversity or variability in the data set. Since in both data sets of fuzzy and crisp harmonic index we have 22 unique values so get similar entropy for both cases.

<table>
<thead>
<tr>
<th>Primes</th>
<th>Crisp Harmonic Index</th>
<th>Fuzzy Harmonic Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>2.47619047619048</td>
<td>10.1789898989899</td>
</tr>
<tr>
<td>11</td>
<td>3.65454545454545</td>
<td>24.5308251717810</td>
</tr>
<tr>
<td>13</td>
<td>4.20512820512821</td>
<td>33.5780230683808</td>
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<td>17</td>
<td>5.27205882352941</td>
<td>55.1354778355809</td>
</tr>
<tr>
<td>19</td>
<td>5.79532163742690</td>
<td>67.5670022740206</td>
</tr>
<tr>
<td>23</td>
<td>6.83003952569170</td>
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<tr>
<td>29</td>
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<td>145.167387770706</td>
</tr>
<tr>
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<td>8.87311827956989</td>
<td>163.63735961643</td>
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<tr>
<td>37</td>
<td>10.3933933933934</td>
<td>224.67853925363</td>
</tr>
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<td>11.4036585365854</td>
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<td>43</td>
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<td>293.931851943150</td>
</tr>
<tr>
<td>47</td>
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<td>344.538365193520</td>
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<tr>
<td>97</td>
<td>25.4589776632302</td>
<td>1256.84413254844</td>
</tr>
</tbody>
</table>

Table 5. Computed Fuzzy Topological Indices
6. Conclusion and opportunity for future work

In conclusion this research paper introduces a set of generalized formulae that are dependent on $\phi$ for both fuzzy and crisp versions of several graph indices. The formulae are applied to the identity graph of the commutative ring $\mathbb{Z}$, and the resulting indices are computed using MATLAB software for 20 prime numbers. The data obtained from these calculations is then utilized for machine learning purposes using Python and Jupyter notebook to explore the correlation between fuzzy and crisp indices. The paper also establishes the connection between fuzzy and crisp indices through the representation of six-degree polynomials and provides an error analysis. The findings of this study significantly contribute to the comprehension of the relationship between fuzzy and crisp graph indices. This knowledge holds potential applications in various fields, particularly in the realms of computer science and Engineering. By establishing generalized formulae and analyzing the relationship between fuzzy and crisp indices, researchers and practitioners can gain deeper insights into the behavior of graph indices in different contexts. This understanding can enhance the development of more efficient algorithms and decision-making processes in various domains.

Additionally it was observed that the accuracy of the results is influenced by the proximity of the input data set. Specifically when the input data set is closer the accuracy of the computed indices improves. This finding underscores the importance of selecting appropriate data sets and emphasizes the need for careful consideration when applying these generalized formulae in practical applications. By taking into account the proximity of the input data set, researchers and practitioners can enhance the accuracy of their calculations and ensure more reliable results. This further strengthens the significance of this research, as it provides valuable insights into the factors that impact the relationship between fuzzy and crisp graph indices. Overall this research not only contributes to the understanding of the relationship between fuzzy and crisp graph indices but also highlights the importance of data quality and proximity in achieving accurate results. These findings have implications for a wide range of fields, offering opportunities for improved decision-making processes and algorithmic advancements in Computer Science and Engineering.

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References


