Well-posedness for Heat Conducting Non-Newtonian Micropolar Fluid Equations

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Abstract

In this paper we consider the first boundary value problem for a class of steady non-Newtonian micropolar fluid equations with heat convection in the three-dimensional smooth and bounded domain Ω. By using fixed-point theorem and introducing a family of penalized problem, under the condition that the external force term and the vortex viscosity coefficient are appropriately small, the existence and uniqueness of strong solutions of the problem are achieved.

Keywords: non-Newtonian fluid; micropolar fluid; heat convection; strong solutions; existence and uniqueness

AMS Subject Classification: 35M33; 35A01; 35D30.

1 Introduction

The motion of an incompressible micropolar fluid with a heat conduction and a constant density is described by the following system of partial differential equations(see [21])

\[
\begin{aligned}
\begin{cases}
\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} - \text{div}\tau + \nabla \pi = 2\nu_r \text{rot} \ \mathbf{\omega} + \mathbf{f}(\theta), \\
\text{div} \mathbf{u} = 0, \\
\mathbf{\omega}_t + (\mathbf{u} \cdot \nabla)\mathbf{\omega} - (c_a + c_d) \Delta \mathbf{\omega} - (c_0 + c_d - c_a) \nabla \text{div} \mathbf{\omega} = 2\nu_r (\text{rot} \mathbf{u} - 2\mathbf{\omega}) + \mathbf{g}(\theta), \\
\theta_t + (\mathbf{u} \cdot \nabla)\theta - \text{div}(\kappa \nabla \theta) = \tau : \mathcal{D} + 4v_r (\frac{1}{2} \text{rot} \mathbf{u} - \mathbf{\omega})^2 + c_0 (\text{div} \mathbf{\omega})^2 \\
\quad + (c_a + c_d) \nabla \mathbf{\omega} : \nabla \omega + (c_d - c_a) \nabla \omega : (\nabla \omega)^T + h.
\end{cases}
\end{aligned}
\]

Equations (1.1) are conservation laws of linear momentum, mass, angular momentum, and energy, respectively. The unknown \( \mathbf{u} = \mathbf{u}(x, t) \) is the velocity vector, \( \pi(x, t) \) the pressure, \( \omega = \omega(x, t) \) the angular velocity of internal rotation of a particle, and \( \theta = \theta(x, t) \) the temperature. The vector-valued functions \( \mathbf{f} \) and \( \mathbf{g} \) denote known external force, scalar-valued function \( h \) the heat source.

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The positive constant $\nu_r$ in (1.1) represents the dynamic micro-rotation viscosity, and $c_0, c_a, c_d$ are constants called coefficients of angular viscosities, $\kappa$ is the heat conductivity. The viscous stress tensor $\tau = \tau(D)$ where

$$D = D(u) = \frac{1}{2}(\nabla u + \nabla u^T)$$

is the rate of deformation tensor, also called the shear rate tensor. If the relation between the stress $\tau$ and the strain rate $D$ is linear then the fluid is called Newtonian. If the relation is non-linear, the fluid is called non-Newtonian. For an introduction to the mechanics of non-Newtonian fluids we refer the reader to references [27, 28].

If $\omega, g$ and the viscosity coefficients $c_0, c_a, c_d, \nu_r$ are zero, system (1.1) reduces to the system of field equations of classical hydrodynamics. In the Newtonian case (i.e. $\tau = \mu D$), several variants of system (1.1) have been studied by several authors in the literature. One well known simplified model is the Oberbeck–Boussinesq approximation which was obtained by ignoring the dissipation term $\tau : D$. The neglect of this term considerably simplifies the analysis and it has been widely studied by several authors from a theoretical interest, we could refer to [3, 7, 11, 23, 29, 31] (and references cited therein) for related results. If the term $\tau : D$ is not neglected, the mathematical analysis to (1.1) becomes significantly more difficult. One of the main challenges stems from the fact that this viscous dissipation term belongs a priori only to $L^1(Q_T)$, which makes the application of compactness arguments problematic. Related results such as the existence, uniqueness, regularity and large time behaviour of solutions have been investigated in previous studies, see e.g. [5, 13, 16, 17, 18, 24, 30] and references therein. In the non-Newtonian case, a popular model is to assume the tensor $\tau$ has $p$-structure. [8] proved the existence of weak solutions to the coupled system of stationary equations (1.1) with the Dirichlet boundary conditions under more general assumptions on $\tau$ with temperature dependent coefficients. [10] proved the existence of a weak solution where $u$ possesses locally integrable second-order derivatives. Under the weak assumptions on data of the problem, [9] proved the existence of weak solutions for a class of non-Newtonian heat-conducting fluids with generalized nonlinear law of heat conduction. [26] has shown the existence of the distributional solution to the steady-state system of equations for non-Newtonian fluid of the $p$-power type coupled with the heat equation with heat sources to have $L^1$-structure and even to be measures. [4] considered the steady flows model with dissipative and adiabatic heating and temperature-dependent material coefficients in a plane bounded domain. The existence of a strong solution is proven by a fixed-point technique based on Schauder theorem for sufficiently small external forces. For more results, we could refer to e.g. [6, 25, 12] and references cited therein.

When the angular momentum balance equation is considered (i.e. including the $\omega$ equation in (1.1)), [19] established the existence and uniqueness of solutions of problem by using the Banach fixed point argument. [1] analyzed the existence, uniqueness, and regularity of the solutions in a bounded domain $\Omega \subset \mathbb{R}^3$ by using an iterative method, the convergence rates in several norms was also considered. [22] studied the stationary problem associated to (1.1), they showed that the boundary value problem has solutions in appropriate Sobolev spaces, provided the viscosities $\nu_r$ and $c_a + c_d$ are sufficiently large. The proof is based on a fixed point argument. The above mentioned results are all regarding the Newtonian case, up to our knowledge, related result for
such a problem of non-Newtonian type has not been considered yet.

In this paper, we study a stationary non-Newtonian version of the full system (1.1) in a smooth bounded domain $\Omega \subset \mathbb{R}^3$. More precisely, by neglecting the dissipation term $\tau : D$, assuming $\tau$ has a $p$-structure

$$\tau(Du) = 2\mu(1 + |Du|^{p-2})Du, \quad \mu > 0 \quad \text{const},$$

after taking the viscosity coefficients properly, we consider the following non-Newtonian micropolar fluid equation with heat convection

$$\begin{cases}
\text{div}(u \otimes u) - \text{div}(Du) + \nabla \pi = 2v_r \text{rot}\omega + \theta f, & \text{in } \Omega, \\
\text{div}u = 0, & \text{in } \Omega, \\
(u \cdot \nabla)\omega - 2\Delta \omega - 2\nabla \text{div}\omega = 2v_r \text{rot}(u - 2\omega) + \theta g, & \text{in } \Omega, \\
-\text{div}(\kappa(\cdot, \theta)\nabla\theta) + (u \cdot \nabla)\theta = \Phi(u, \omega) + h, & \text{in } \Omega,
\end{cases}$$

supplemented with the first boundary value conditions

$$u|_{\partial\Omega} = 0, \quad \omega|_{\partial\Omega} = 0, \quad \theta|_{\partial\Omega} = 0. \quad (1.4)$$

Where in (1.3), $\Phi(u, \omega) = \sum_{i=1}^4 \Phi_i$ and

$$\Phi_1(u, \omega) = \left(\frac{1}{2} \text{rot}u - \omega\right)^2, \quad \Phi_2(\omega) = (\text{div}\omega)^2, \quad \Phi_3(\omega) = 2 \sum_{i,j=1}^3 (\omega_{i,j})^2, \quad \Phi_4(\omega) = \sum_{i,j=1}^3 \omega_{i,j}\omega_{j,i},$$

here for a vector valued functions $v(x)$, we denote $v_{i,j} = \partial_j v_i(x)$. We assume that the heat conductivity $\kappa : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a $C^1$ function such that $0 < \kappa_1 \leq \kappa(x, \theta) \leq \kappa_2$ a.e. $x \in \Omega$ and for all $\theta \in \mathbb{R}$ it satisfies $|\kappa'(\cdot, a) - \kappa'(\cdot, b)| \leq \lambda' |a - b|$ for all $a, b \in \mathbb{R}$ and $\kappa'(\cdot, 0) = 0$, where $\kappa_1, \kappa_2$ and $\lambda'$ are positive constants.

The goal of the present paper is to prove the existence and uniqueness of a strong solution to system (1.2)-(1.4) under a smallness condition on the external force term and the vortex viscosity coefficient. The procedure employs similar ideas to the ones presented in [2]. The main idea is to use the fixed-point theorem combined with the regularized technique.

Let us briefly sketch the proof. First, after regularizing the term $|D(u)|$ in the stress tensor with a parameter $\varepsilon$, we consider a penalized problem and rewrite it in a new form. Next, by the known results about linear equation, we define the mapping by linearizing the above systems. Noticing that, the first equation of the linearized systems is in a form of Stokes type, by using the well known regularity resuls (see [14]), we could obtain a pair $(u_\varepsilon, \pi_\varepsilon) \in W^{2,q}(\Omega) \times W^{1,q}(\Omega)$. What needs to be pointed out is that if we do not regularize the stress tensor, the right hand side of this equation does not belong to $L^q(\Omega)$, this makes it impossible to apply the above theorem to get $u_\varepsilon \in W^{2,q}(\Omega)$. Then, by using the fixed-point theorem, we could prove the existence of an approximate solutions $(u_\varepsilon, \omega_\varepsilon, \theta_\varepsilon)$, and finally by taking $\varepsilon \to 0$, we prove the main result (Theorem 2.1).

**Remark 1.1** In our case, in the process of proof we used an elementary inequality: for every $a, b \in \mathbb{R}^+$, there have

$$|(1 + a)^{p-2} - (1 + b)^{p-2}| \leq (|p - 2|, 1)^+ |a - b| (1 + (a, b)^+(p-2)^{-1})^{|p - 2| - 1}.$$
If we allow the stress tensor to be with singularity (i.e. \( \tau(Du) = 2\mu|Du|^{(p-2)}Du \)), one similar estimate for \( |a^{p-2} - b^{p-2}| \) is needed and this is not known. Therefore, our method is not suitable for the singular case. (See [2] for more details).

The paper is organized as follows. In Section 2, we introduce basic notations and some preliminary results that will be used later, then state the main results of this work. We prove the existence and uniqueness of strong solutions of an approximate problem to (1.2)-(1.4) in section 3 by a fixed point argument. Finally, in section 4 we prove the main result by letting the parameter \( \varepsilon \to 0 \).

## 2 Preliminaries and main result

Throughout the paper, we shall use the following functional spaces: \( L^q(\Omega) \), \( W^{m,q}(\Omega) \), \( W_0^{1,q}(\Omega) \) are usual Lebesgue and Sobolev spaces; the norms in \( L^q(\Omega) \) and \( W^{m,q}(\Omega) \) we denote by \( \| \cdot \|_q \) and \( \| \cdot \|_{m,q} \); \( W^{-1,q}(\Omega) \) denotes the dual space of \( W_0^{1,q}(\Omega) \), and its norm is represented by \( \| \cdot \|_{-1,q;\Omega} \).

We also introduce the space

\[
\mathcal{V} := \{ u : u \in C_0^\infty(\Omega), \text{ div } u = 0 \};
\]

\[
V_p := \{ u \in W_0^{1,p}(\Omega) : \text{div} u = 0 \};
\]

\[
V_{m,p} := \{ u \in W_0^{1,p}(\Omega) \cap W^{m,p}(\Omega) : \text{div} u = 0, \text{ in } \Omega \}.
\]

For \( x, y \in \mathbb{R} \), \((x, y)^+ = \max\{x, y\}, x^+ = \max\{x, 0\} \), \( S_p = ([p-2], 2)^+ \). Introduce constants

\[
2r_p = 1 + (p-3)^+ - (p-4)^+, \quad \gamma_p = \frac{[(p, 3)^+ - 2][p, 3)^+ - 2}{[(p, 3)^+ - 1][p, 3)^+ - 1},
\]

we also denote \( C_p = C_p(n, s, \Omega) \) the Poincaré constant of Poincaré inequality.

For \( q > r > s > 3 \), let us consider the convex set \( B_\rho \) defined by

\[
B_\rho = \{ (\xi, \eta, \zeta) \in V_{2,q} \times W^{2,r}(\Omega) \times W^{2,s}(\Omega) : C_E\|\nabla \xi\|_{1,q} \leq \rho, \ C_E\|\nabla \eta\|_{1,r} \leq \rho, \ C_E\|\nabla \zeta\|_{1,s} \leq \rho \},
\]

(2.2)

where \( \rho \) is a constant to be determined; \( C_E, C_E, C_{\Omega} \) are the embedding constants from \( W^{1,q}(\Omega) \) into \( L^\infty(\Omega) \), \( W^{1,r}(\Omega) \) into \( L^\infty(\Omega) \), \( W^{1,s}(\Omega) \) into \( L^\infty(\Omega) \), respectively. Also, we consider the space \( V_{2,q} \times (W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)) \times (W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega)) \) endowed with the norm

\[
\|(\xi, \eta, \zeta)\|_{1,q,r,s} := \max\{\|\nabla \xi\|_{1,q}, \|\nabla \eta\|_{1,r}, \|\nabla \zeta\|_{1,s}\}.
\]

For latter use, we state some useful Lemmas.

**Lemma 2.1** ([14]) Let \( m \geq -1 \) be an integer and let \( \Omega \) be a bounded in \( \mathbb{R}^n(n = 2, 3) \) with boundary \( \partial \Omega \) of class \( C^k \) with \( k = (m + 2) \). Then for any \( f \in W^{m,q}(\Omega) \), the following system

\[
\begin{cases}
-\Delta u + \nabla \pi = f, & \text{in } \Omega, \\
\text{div } u = 0, & \text{in } \Omega, \\
u_{|\partial \Omega} = 0, &
\end{cases}
\]

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admits a unique \((u, \pi) \in W^{m+2,q}(\Omega) \times W^{m+1,q}(\Omega)\). Moreover, the following estimate holds
\[ \|\nabla u\|_{m+1,q} + \|\pi\|_{m+1,q/\mathbb{R}} \leq C_m\|f\|_{m,q}, \]
where \(C_m = C_m(n,q,\Omega)\) is a positive constant.

**Lemma 2.2** \(^{[2]}\) Let \(\gamma_p\) be defined as (2.1), and let \(L: \mathbb{R}^+ \rightarrow \mathbb{R}\) be defined by
\[ L(\delta) = A\delta^2 - \delta + E\delta\ell(\delta) + D, \]
where \(A, D, E\) are positive constants and \(\ell(x) = x(1 + x)^{(p-3)^+}\). Thus, if the following assertion holds
\[ AD + ED(1 + D)^{(p-3)^+} \leq \gamma_p, \]
then \(L\) possesses at least one root \(\delta_1\). Moreover, \(\delta_1 > D\) and for every \(\beta \in [1,2]\) the following estimate holds
\[ \frac{\beta - 1}{\beta}\delta_1 + \frac{2 - \beta}{\beta} A\delta_1^2 + \frac{2 - \beta}{\beta} E\delta_1\ell(\delta_1) + \frac{E(p - 3)^+}{\beta}\delta_1^3(1 + \delta_1)^{(p-3)^+ - 1} \leq D. \tag{2.3} \]

**Lemma 2.3** \(^{[20]}\) Let \(X\) and \(Y\) be Banach spaces such that \(X\) is reflexive and \(X \hookrightarrow Y\). Let \(B\) be a non-empty, closed, convex and bounded subset of \(X\) and let \(T: B \rightarrow B\) be a mapping such that
\[ \|T(u) - T(v)\|_Y \leq K\|u - v\|_Y, \quad \text{for each } u, v \in B, \quad 0 < K < 1, \]
then \(T\) has a unique fixed point in \(B\).

The main result of our paper is the following:

**Theorem 2.1** Let \(f \in L^q(\Omega), g \in L^r(\Omega), h \in L^s(\Omega)\), where \(q > r > s > 3, p > 1, \mu > 0\). There exist a positive constant \(\lambda = \lambda(C_0, C_{-1}, C_3, \lambda', \kappa, C_E, C_E, C_p, m_1 = m_1(C_1, C_p, C_E, C_E, \nu r), m_2 = m_2(C, \lambda, C_p, C_E, C_E, C_E, C_E),\) such that if \(\|g\|_r < m_1, \|h\|_s < m_2k^2, \nu r > 0\) small enough, and
\[ \left(1 + 2\frac{2\lambda^2\|f\|_q^2 + \lambda\nu r}{\mu}\right) + \left(1 + \frac{1}{\mu}\right)\lambda\nu r + \lambda\|f\|_q + \lambda\|g\|_r \]
\[ + \frac{\lambda^2}{\mu}\|f\|_q^2 + \lambda\nu r \left(1 + \frac{\lambda(\|f\|_q^2 + \lambda\nu r)}{\mu}\right)^{(p-3)^+} < \frac{1}{4(p-2,1)^++} \tag{2.4} \]
then problem (1.2)-(1.4) has a unique strong solution \((u, \omega, \theta) \in V_{2,q} \times (W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)) \times (W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega)).\)

**Remark 2.1** As usual, the pressure \(\pi\) has disappeared from the notion of solution. Actually, the pressure may be recovered by de Rham Theorem at least in \(L^2(\Omega)\), such that the \((u, \pi, \omega, \theta)\) satisfies equations (1.2)-(1.4) almost everywhere (see e.g. [15]).
3 Existence of the approximate solution

For $0 < \varepsilon < 1$, we consider the family of the penalized problem

$$
\begin{cases}
-\text{div}(2\mu(1 + \sqrt{\varepsilon^2 + |D\mathbf{u}|^2})^{p-2}D\mathbf{u}) + \nabla \pi + \text{div}(\mathbf{u} \otimes \mathbf{u}) = 2\nu_r \text{rot} \mathbf{u} + \theta \mathbf{f}, & \text{in } \Omega, \\
\text{div} \mathbf{u} = 0, & \text{in } \Omega, \\
-2\Delta \omega + (\mathbf{u} \cdot \nabla)\omega - 2\nu_r \text{div}\mathbf{u} - 2\omega + \theta \mathbf{g}, & \text{in } \Omega, \\
-\text{div}(\kappa(\cdot, \theta)\nabla \theta) + (\mathbf{u} \cdot \nabla)\theta = \Phi(\mathbf{u}, \omega) + h, & \text{in } \Omega, \\
\mathbf{u}|_{\partial \Omega} = 0, & \omega|_{\partial \Omega} = 0, \quad \theta|_{\partial \Omega} = 0.
\end{cases}
$$

(3.1)

The following result holds true.

**Theorem 3.1** Let $\mathbf{f} \in L^q(\Omega)$, $\mathbf{g} \in L^r(\Omega)$, $h \in L^s(\Omega)$, where $q > r > s > 3$, $p > 1$, $\mu > 0$, and $0 < \varepsilon < 1$. There exist a positive constant $\lambda = \lambda(C_0, C_{-1}, C_3, \lambda', \kappa_1, C_E, C_{E'}, C_p)$, $m_1 = m_1(C_1, C_p, C_E, C_{E'}, \nu_r)$, $m_2 = m_2(C, \lambda', C_p, C_E, C_{E'}, C_p)$, such that if $\|\mathbf{g}\|_r < m_1$, $\|h\|_s < m_2 \kappa_1^2$, $\nu_r > 0$ is small enough, and

$$
\left(\frac{1}{\mu} + 2\right) \frac{2\lambda^2}{\mu}\left(\|\mathbf{f}\|_q^2 + \nu_r\right) + \frac{1}{\mu} + 1\lambda \nu_r + \lambda \|\mathbf{f}\|_q + \lambda \|\mathbf{g}\|_r
+ 3p\lambda^2 \frac{\|\mathbf{f}\|_q^2 + \nu_r}{\mu} \left(1 + \frac{\lambda\|\mathbf{f}\|_q^2 + \nu_r}{\mu}\right)^{(p-3)^+} < \frac{1}{4(p-2,1)^p},
$$

(3.2)

then problem (3.1) has a unique strong solution

$$(\mathbf{u}_\varepsilon, \omega_\varepsilon, \theta_\varepsilon) \in V_{2,q} \times (W^{2,r}(\Omega) \cap W^{1,r}_0(\Omega)) \times (W^{2,s}(\Omega) \cap W^{1,s}_0(\Omega)).$$

**Proof.** We use a fixed point argument to prove Theorem 3.1, and the proof will be divided into four steps.

**Step 1: Linearization of the problem and construction of the mapping.**

Reformulate the problem (3.1) as

$$
\begin{cases}
-\mu(1 + \varepsilon)^{(p-2)} \Delta \mathbf{u} + \nabla \pi = 2\nu_r \text{rot} \mathbf{u} + \theta \mathbf{f} - \text{div}(\mathbf{u} \otimes \mathbf{u}) + \text{div}(2\mu\sigma_\varepsilon(|D\mathbf{u}|^2)D\mathbf{u}), \\
\text{div} \mathbf{u} = 0, \\
-2\Delta \omega - 2\nu_r \text{div}\mathbf{u} = 2\nu_r \text{rot} \mathbf{u} + \theta \mathbf{g} - 4\nu_r \omega - (\mathbf{u} \cdot \nabla)\omega, \\
-\kappa(\cdot, \theta)\Delta \theta = \kappa'(\cdot, \theta)\nabla \theta^2 - (\mathbf{u} \cdot \nabla)\theta + \Phi(\mathbf{u}, \omega) + h, \\
\mathbf{u}|_{\partial \Omega} = 0, & \omega|_{\partial \Omega} = 0, \quad \theta|_{\partial \Omega} = 0,
\end{cases}
$$

(3.3)

where $\sigma_\varepsilon(x^2) = (1 + \sqrt{\varepsilon^2 + |x|^2})^{(p-2)} - (1 + \varepsilon)^{(p-2)}$.

We define the operator

$$
T_\varepsilon : V_{2,q} \times (W^{2,r}(\Omega) \cap W^{1,r}_0(\Omega)) \times (W^{2,s}(\Omega) \cap W^{1,s}_0(\Omega)) \rightarrow V_{2,q} \times (W^{2,r}(\Omega) \cap W^{1,r}_0(\Omega)) \times (W^{2,s}(\Omega) \cap W^{1,s}_0(\Omega)),
$$

as...
given by $T_\varepsilon(\xi, \eta, \zeta) = (u_\varepsilon, \omega_\varepsilon, \theta_\varepsilon)$, where $(u_\varepsilon, \omega_\varepsilon, \theta_\varepsilon)$ is the solution of the problem

\[
\begin{align*}
-\mu(1 + \varepsilon)(p-2)\Delta u_\varepsilon + \nabla \pi_\varepsilon &= 2\nu_r \text{rot } \eta + \zeta f - \text{div}(\xi \otimes \xi) + \text{div}(2\mu\sigma_\varepsilon(|D\xi|^2)D\xi), \\
\text{div } u_\varepsilon &= 0, \\
-2\Delta \omega_\varepsilon - 2\nabla \text{div } \omega_\varepsilon &= 2\nu_r \text{rot } \xi + \zeta g - 4\nu_r \eta - (\xi \cdot \nabla) \eta, \\
-\kappa(\cdot, \theta_\varepsilon) \Delta \theta_\varepsilon &= \kappa'(\cdot, \zeta) |\nabla \zeta|^2 - (\xi \cdot \nabla) \xi + \Phi(\xi, \eta) + h, \\
u_e|_{\partial \Omega} &= 0, \quad \omega_e|_{\partial \Omega} = 0, \quad \theta_e|_{\partial \Omega} = 0.
\end{align*}
\]

**Step 2:** Proving $T_\varepsilon$ maps $B_\rho$ onto itself.

In this part, we will prove that there exists a constant $\rho > 0$, such that $T_\varepsilon$ maps $B_\rho$ onto $B_\rho$. We formulate the result as follows.

**Proposition 3.1** Let $f \in L^q(\Omega)$, $g \in L^r(\Omega)$, where $q > r > s > 3$, $p > 1$, $\mu > 0$. There exist positive constants $\overline{\lambda}_1 = \overline{\lambda}_1(C_0, C_\rho, C_E, C_F)$, $m_1 = m_1(C_1, C_E, C_F, C, \nu_r)$ and $m_2 = m_2(C_1, C_E, C_F, C)$ such that if $\|g\|_r < m_1$, $\|h\|_s < m_2$ and $\nu_r > 0$ is small enough, and

\[
\frac{\lambda_1^2 (\mu^2 q + \nu_r)}{\mu^2} + \frac{\lambda_2 q^2 p}{\mu} \left( \frac{1}{\nu_r} \right) \left( 1 + \frac{\lambda_1 (\mu^2 q + \nu_r)}{\mu} \right) \leq \gamma_p,
\]

then $T_\varepsilon(B_\rho) \subseteq B_\rho$ for some $\rho > 0$.

**Proof.** Let $(\xi, \eta, \zeta)$ be in $B_\rho$ (see 2.2). Using Lemma 2.1, we obtain that $u_\varepsilon \in V_{2,q}$ and

\[
\|\nabla u_\varepsilon\|_{1,q} \leq \frac{C_0}{(1 + \varepsilon)(p-2)} \left( \|\xi\|_q + \|\xi \cdot \nabla \xi\|_q + \|2\nu_r \text{rot } \eta\|_q + \|\text{div}[2\nu_r \sigma_\varepsilon(|D\xi|^2)D\xi]\|_q \right). \tag{3.6}
\]

Firstly, we have

\[
\|2\nu_r \text{rot } \eta\|_q \leq 2\nu_r C \|\nabla \eta\|_q \leq C \nu_r \|\nabla \xi\|_{1,r} \leq \frac{C \nu_r}{C_E} \rho^2. \tag{3.7}
\]

\[
\|\xi \cdot \nabla \xi\|_q \leq \|\nabla \xi\|_q \leq C_E(C_p + 1) \|\nabla \xi\|_q \leq \rho(C_p + 1) \|\xi\|_q \leq \frac{\rho^2 (C_p + 1)^2}{2} + \frac{\|\xi\|_q^2}{2}. \tag{3.8}
\]

Reasoning as in [2], we obtain

\[
\|\xi \cdot \nabla \xi\|_q \leq \|\xi \cdot \nabla \xi\|_q + \|\text{div}[2\nu_r \sigma_\varepsilon(|D\xi|^2)D\xi]\|_q \leq \frac{C_p}{C_E} \rho^2 + \frac{8 \mu \lambda_p}{C_E} \rho \ell(\rho). \tag{3.9}
\]

Combining (3.6) to (3.9), we conclude that

\[
\|\nabla u_\varepsilon\|_{1,q} \leq \frac{C_0}{\mu} \left( \frac{\|\xi\|_q^2}{2} + \frac{C \nu_r}{C_E} \rho^2 + \left( \frac{(C_p + 1)^2}{2} + \frac{C_p}{C_E} \rho^2 + \frac{8 \mu \lambda_p}{C_E} \rho \ell(\rho) \right) \right) \leq \frac{\lambda_1^2}{\mu} \left( \frac{\|\xi\|_q^2}{2} + \nu_r \rho^2 + \mu \lambda_p \rho \ell(\rho) \right), \tag{3.10}
\]

where $\lambda_1 = C_0 \max \left\{ \frac{1}{2}, \frac{C_p}{C_E}, \frac{(C_p + 1)^2}{2}, \frac{8 \mu \lambda_p}{C_E} \right\}$, $\ell(x) = x(1 + x)^{(p-3)+}$, $\lambda_p = (|p-2|, 1)^{2(p-3)}$. 

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On the other hand, by the theory of elliptic equations, there is a positive constant $C_1$ such that

$$\| \nabla \omega \|_{1,r} \leq C_1 (\| \xi \|_{r} + \| 2 \nu \cdot \mathrm{rot} \xi \|_{r} + \| 4 \nu \cdot \eta \|_{r} + \| \xi \cdot \nabla \eta \|_{r})$$

$$\leq C_1 (\| \xi \|_{\infty} \| g \|_{r} + 2 \nu \cdot C_{\nabla} \| \nabla \xi \|_{r} + 4 \nu \cdot C_{P} \| \nabla \eta \|_{r} + \| \xi \|_{\infty} \| \nabla \eta \|_{r})$$

$$\leq C_1 [C_{\nabla}(C_p + 1) \| \nabla \xi \|_{1,q} + C_{\nu} \| \nabla \xi \|_{1,q} + 4 \nu \cdot C_{P} \| \nabla \eta \|_{1,r} + C_E \| \xi \|_{1,q} \| \nabla \eta \|_{1,r}]$$

$$\leq C_1 \rho(C_p + 1) \| g \|_{r} + C_{\nu} \| \nabla \xi \|_{1,q} + 4 \nu \cdot C_{P} \| \nabla \eta \|_{1,r} + C_E(C_p + 1) \| \nabla \xi \|_{q} \| \nabla \eta \|_{1,r}$$

$$\leq C_1 \left[ \frac{|g|^2}{2} + \frac{\rho^2(C_p + 1)^2}{2} + C_{\nu} \| \nabla \xi \|_{1,q} + 4 \nu \cdot C_{P} \| \nabla \eta \|_{1,r} + C_E(C_p + 1) \| \nabla \xi \|_{1,q} \frac{\rho}{C_E} \right]$$

$$\leq \frac{C_1(C_p + 1)(C_p + 1)C_E + 2}{2C_E} \rho^2 + 2 \lambda \nu \rho + \frac{C_1}{2} |g|^2.$$  \hspace{1cm} (3.11)

where $\lambda_2 = C_1 \max \{ \frac{C_E}{C_E}, \frac{4C_p}{C_E} \}$.

Also, from the elliptic equation (3.4), there exists a positive constant $C_2$ such that

$$\| \nabla \eta \|_{1,s} \leq \frac{C_2}{\kappa_1} \| \kappa(\cdot, \cdot) \| \nabla \xi \|_{s} + \frac{C_2}{\kappa_1} \| \Phi(\xi, \eta) \|_{s} + \frac{C_2}{\kappa_1} \| \xi \cdot \nabla \zeta \|_{s} + \frac{C_2}{\kappa_1} \| h \|_{s}$$

$$= \frac{C_2}{\kappa_1} \| \kappa(\cdot, \cdot) \| \nabla \xi \|_{s} + \frac{C_2}{\kappa_1} \sum_{i=1}^{4} \Phi_{i}(\xi, \eta) \|_{s} + \frac{C_2}{\kappa_1} \| \xi \cdot \nabla \zeta \|_{s} + \frac{C_2}{\kappa_1} \| h \|_{s}. \hspace{1cm} (3.12)$$

By the assumptions of $\kappa(\cdot, \theta)$, it follows that

$$\| \kappa(\cdot, \zeta) \| \nabla \xi \|_{s} = \| (\kappa(\cdot, \zeta) - \kappa(\cdot, 0)) \| \nabla \xi \|_{s} \leq \lambda \| \xi \|_{\infty} \| \nabla \xi \|_{2s}$$

$$\leq \lambda C_{\nabla}(C_p + 1) \| \nabla \xi \|_{s} \| \nabla \xi \|_{2s} \leq \frac{\lambda C}{C_{E}^2} \rho^2; \hspace{1cm} (3.13)$$

Since

$$\| \Phi_{1}(\xi, \eta) \|_{s} = \| (\frac{1}{2} \mathrm{rot} \xi - \eta)^2 \|_{s} \leq C \left( \int_{\Omega} (\nabla \xi)^2 + \eta^2 \right)^{1/2} \leq C(\| \nabla \xi \|_{2s}^2 + \| \eta \|_{2s}^2),$$

and

$$\| \Phi_{2}(\eta) \|_{s} + \| \Phi_{3}(\eta) \|_{s} + \| \Phi_{4}(\eta) \|_{s}$$

$$\leq \| (\mathrm{div} \eta)^2 \|_{s} + \| [3 \sum_{i=1}^{3} (\eta_{i,j})^2] \|_{s} + \| [3 \sum_{i=1}^{3} \eta_{i,j} \eta_{j,i}] \|_{s}$$

$$\leq C \left( \int_{\Omega} (\nabla \eta)^2 \right)^{1/2} + C \left( \int_{\Omega} (\nabla \eta)^2 \right)^{1/2} + C \left( \int_{\Omega} (\nabla \eta)^2 \right)^{1/2}$$

$$\leq C \| \nabla \eta \|_{2s}^2,$$

whence

$$\| \Phi(\xi, \eta) \|_{s} \leq C(\| \nabla \xi \|_{2s}^2 + \| \nabla \eta \|_{2s}^2) \leq C(\| \nabla \xi \|_{1,q}^2 + \| \nabla \eta \|_{1,r}^2) \leq \frac{C p^2}{C_E^2} + \frac{C p^2}{C_E^2}. \hspace{1cm} (3.14)$$
Finally
\[ \| \xi \cdot \nabla \zeta \|_s \leq \| \xi \|_\infty \| \nabla \zeta \|_s \leq C_{E_1} \| \xi \|_1, q \| \nabla \zeta \|_s \leq (C_p + 1) C_{E_1} \| \nabla \xi \|_q \| \nabla \zeta \|_s \leq \frac{(C_p + 1) C_{E_1}}{\rho^2}. \]  
(3.15)

Combining (3.12) to (3.15), we obtain
\[ \| \nabla \theta \|_{1, s} \leq \frac{C_2 \lambda' C(C_p + 1)}{\kappa_1 C_{E_1}^2} \rho^3 + \frac{C_2(C_p + 1)}{\kappa_1 C_{E_1}} \rho^2 + \frac{C_2}{\kappa_1} \rho \| h \|_s \]
\[ \leq \frac{C_2 \lambda' C(C_p + 1)}{\kappa_1 C_{E_1}^2} \rho^3 + 2 \frac{\lambda_3}{\lambda_1} \rho^2 + \frac{C_2}{\kappa_1} \rho \| h \|_s, \]
(3.16)

where \( \lambda_3 = \max \left\{ \frac{C_{E_1} + 1}{C_{E_1}}, \frac{C_{E_1}}{C_{E_1}^2} + \frac{C_{E_1}}{C_{E_1}} \right\} \).

Without loss of generality, it can be assumed that \( \rho \leq 1 \). To ensure that \( T_\varepsilon(B_\rho) \subseteq B_\rho \), it is sufficient to require
\[ \| \nabla u_\varepsilon \|_{1, q} \leq \frac{\lambda_1}{\mu} \left[ \| f \|_q^2 + \nu_r \rho + \rho^2 + \mu S_p \rho \ell(\rho) \right] \leq \frac{\lambda_1}{\mu} \left[ \| f \|_q^2 + \nu_r + \rho^2 + \mu S_p \rho \ell(\rho) \right] \leq \rho; \]  
(3.17)
\[ \| \nabla \omega_\varepsilon \|_{1, r} \leq \frac{C_1(C_p + 1)[(C_p + 1)C_{E_1} + 2]}{2C_{E_1}^2} \rho^2 + 2 \lambda_2 \nu_r + \rho + \frac{C_1}{2} \| g \|_r^2 \leq \rho; \]  
(3.18)
\[ \| \nabla \theta \|_{1, s} \leq \frac{C_2 \lambda' C(C_p + 1)}{\kappa_1 C_{E_1}^2} \rho^3 + 2 \frac{\lambda_3}{\kappa_1} \rho^2 + \frac{C_2}{\kappa_1} \rho \| h \|_s \leq \left( \frac{C_2 \lambda' C(C_p + 1)}{\kappa_1 C_{E_1}^2} + 2 \frac{\lambda_3}{\kappa_1} \right) \rho^3 + \frac{C_2}{\kappa_1} \rho \| h \|_s \leq \rho. \]  
(3.19)

Applying Lemma 2.2 with \( A = \frac{\lambda_1}{\mu}, E = \lambda_1 S_p \) and \( D = \frac{\lambda_1(\| f \|_q^2 + \nu_r)}{\mu} \), there exists \( \rho_1 > \frac{\lambda_1(\| f \|_q^2 + \nu_r)}{\mu} \) such that
\[ \frac{\lambda_1}{\mu} \left[ \| f \|_q^2 + \nu_r + \rho_1^2 + \mu S_p \rho_1 \ell(\rho_1) \right] \leq \rho_1, \]
moreover, by taking \( \beta = 2 \) in (2.3), we have
\[ \rho_1 \leq \frac{2 \lambda_1(\| f \|_q^2 + \nu_r)}{\mu}. \]

Reformulate inequality (3.18) as
\[ \frac{C_1(C_p + 1)[(C_p + 1)C_{E_1} + 2]}{2C_{E_1}^2} \rho^2 + (2 \lambda_2 \nu_r - 1) \rho + \frac{C_1}{2} \| g \|_r^2 \leq 0, \]
(3.20)
since the discriminant \( \Delta = (2 \lambda_2 \nu_r - 1)^2 - \frac{C_1^2(C_p + 1)(C_p + 1)C_{E_1}^2}{C_{E_1}^2} \| g \|_r^2 > 0 \), namely
\[ \| g \|_r^2 \leq \frac{(2 \lambda_2 \nu_r - 1)^2 C_{E_1}}{C_1^2(C_p + 1)(C_p + 1)C_{E_1}^2} \equiv m_1, \]
we deduce that the inequality (3.20) is valid for some \( \rho \).
Take a constant $D$ satisfying $\rho^-_2 < D < \rho^+_2$, where

$$\rho^\pm_2 = \frac{C_E}{C_1(C_p + 1)((C_p + 1)C_E + 2)} \cdot \left[ (1 - 2\lambda_2 \nu_r) \pm \sqrt{(2\lambda_2 \nu_r - 1)^2 - \frac{C_2^2(C_p + 1)((C_p + 1)C_E + 2)}{C_E}} \right]$$

$$= \frac{C_1 m_1}{1 - 2\lambda_2 \nu_r} \left[ 1 \pm \sqrt{1 - \frac{\|g\|^2}{m_1}} \right],$$

since for every $\rho \in [\rho^-_2, \rho^+_2]$, inequality (3.20) holds true, we could choose $\rho_2 \in (\rho^+_2, D)$ such that

$$\frac{C_1(C_p + 1)((C_p + 1)C_E + 2)}{2C_E} \rho_2^2 + 2\lambda_2 \nu_r \rho_2 + \frac{C_1}{2} \|g\|^2 \leq \rho_2.$$

On the other hand, we rewrite inequality (3.19) as

$$\left( \frac{C_2 \lambda' C(C_p + 1)}{\kappa_1 C_E^2} + 2\frac{\lambda_3}{\kappa_1} \right) \rho^2 - \rho + \frac{C_2}{\kappa_1} \|h\|_s \leq 0,$$

(3.21)

since $\Delta = 1 - 4 \left( \frac{C_2 \lambda' C(C_p + 1)}{\kappa_1 C_E^2} + 2\frac{\lambda_3}{\kappa_1} \right) \frac{\|h\|_s}{\kappa_1} > 0$, namely

$$\frac{\|h\|_s}{k_1} < \frac{C_2^2}{4C_2 |C_2 \lambda' C(C_p + 1) + 2\lambda_3 C_E^2|} \equiv m_2,$$

it follows that inequality (3.21) is valid for some $\rho$.

The above $D$ could be also selected to satisfy $\rho^-_3 < 2D < \rho^+_3$, where

$$\rho^\pm_3 = \frac{\kappa_1 C_E^2}{2[C_2 \lambda' C(C_p + 1) + 2\lambda_3 C_E^2]} \left[ 1 \pm \sqrt{1 - 4 \left( \frac{C_2 \lambda' C(C_p + 1)}{\kappa_1 C_E^2} + 2\frac{\lambda_3}{\kappa_1} \right) \frac{\|h\|_s}{\kappa_1}} \right]$$

$$= \frac{2m_2}{\kappa_1} \left[ 1 \pm \sqrt{1 - \frac{\|h\|_s}{k_1^2 m_2}} \right],$$

since (3.21) is valid for every $\rho \in [\rho^-_3, \rho^+_3]$, we can choose $\rho_3 \in (2D, \rho^+_3)$ such that

$$\left( \frac{C_2 \lambda' C(C_p + 1)}{\kappa_1 C_E^2} + 2\frac{\lambda_3}{\kappa_1} \right) \rho_3^2 + \frac{C_2}{\kappa_1} \|h\|_s \leq \rho_3.$$

In conclusion, we have obtained

$$\rho_2 < \frac{\lambda_1 (\|f\|^2_q + \nu_r)}{\mu} < \rho_1 < \frac{2 \lambda_1 (\|f\|^2_q + \nu_r)}{\mu} < \rho_3,$$

(3.22)

which completes the proof by taking $\rho = \rho_1$.

**Step 3: Proving $T_\varepsilon : B_\rho \to B_\rho$ is a contraction.**

In this step, we concentrate on proving the map $T_\varepsilon : B_\rho \to B_\rho$ is a contraction. Our aim is to prove the following result.
Proposition 3.2 There is a positive constant $\overline{\lambda}_0 = \overline{\lambda}_0(C_{-1}, C_3, X, \lambda, \kappa_1, C_p, C_E, C_\overline{E}, C_\overline{\overline{E}})$ such that if

$$\overline{\lambda}_0 \left[ \left( \frac{1}{\mu} + 2 \right) \frac{2 \lambda_1(||f||_q^2 + \nu_r)}{\mu} + \left( \frac{1}{\mu} + 1 \right) \nu_r + \frac{||f||_q}{\mu} + ||g||_r, \right. \left. + \sum_p \overline{\lambda}_1(||f||_q^2 + \nu_r) \left( 1 + \overline{\lambda}_1(||f||_q^2 + \nu_r) \right)^{(p-3)^+} \right] < \frac{1}{4(\mu-2.1)^+}.$$  

then $T_\varepsilon : B_\rho \to B_\rho$ is a contraction in $W^{1,q}_0(\Omega) \times W^{1,r}_0(\Omega)$. 

Proof. Let $(\xi, \eta, \zeta), (\tilde{\xi}, \tilde{\eta}, \tilde{\zeta}) \in B_\rho$, and let $[u_\varepsilon, \omega, \theta], [\tilde{u}_\varepsilon, \tilde{\omega}, \tilde{\theta}]$ be their respective images under $T_\varepsilon$. Then, from (3.4) we obtain

$$\begin{cases}
-\mu(1 + \varepsilon)^{(p-2)} \triangle(u_\varepsilon - \tilde{u}_\varepsilon) + \nabla(p_\varepsilon - \tilde{p}_\varepsilon) = F_\varepsilon, \\
\text{div}(u_\varepsilon - \tilde{u}_\varepsilon) = 0, \\
-2\triangle(\omega_\varepsilon - \tilde{\omega}_\varepsilon) - 2\nabla(\omega_\varepsilon - \tilde{\omega}_\varepsilon) = G, \\
-\kappa(\cdot, \theta_\varepsilon) \triangle \theta_\varepsilon + \kappa(\cdot, \tilde{\theta}_\varepsilon) \triangle \tilde{\theta}_\varepsilon = H + \kappa(\cdot, \zeta)(\nabla \zeta)^2 - \kappa(\cdot, \tilde{\zeta})(\nabla \tilde{\zeta})^2, \\
(u_\varepsilon - \tilde{u}_\varepsilon)_{|_{\partial \Omega}} = 0, \quad (\omega_\varepsilon - \tilde{\omega}_\varepsilon)_{|_{\partial \Omega}} = 0, \quad (\theta_\varepsilon - \tilde{\theta}_\varepsilon)_{|_{\partial \Omega}} = 0,
\end{cases}$$  

(3.24)

where

$$F_\varepsilon := \text{div}(\tilde{\xi} \otimes \xi - \xi \otimes \tilde{\xi}) + 2\nu_\varepsilon \text{rot}(\eta - \tilde{\eta}) + 2\mu \text{div}[\sigma_\varepsilon(||D\xi||^2)D\xi - \sigma_\varepsilon(||D\tilde{\xi}||^2)D\tilde{\xi}] + (\zeta - \tilde{\zeta})f;$$

$$G := 2\nu_\varepsilon \text{rot}(\xi - \tilde{\xi}) - 4\nu_\varepsilon(\eta - \tilde{\eta}) - (\xi \cdot \nabla)\eta + (\tilde{\xi} \cdot \nabla)\tilde{\eta} + (\zeta - \tilde{\zeta})g;$$

$$H := \Phi(\xi, \eta) - \Phi(\tilde{\xi}, \tilde{\eta}) - (\xi \cdot \nabla)\zeta + (\tilde{\xi} \cdot \nabla)\tilde{\zeta}.$$

From Lemma 2.1, we obtain

$$||\nabla(u_\varepsilon - \tilde{u}_\varepsilon)||_q \leq \frac{C_{-1}}{\mu} ||\text{div}(\tilde{\xi} \otimes \xi - \xi \otimes \tilde{\xi})||_{-1,q} + 2\nu_\varepsilon ||\text{rot}(\eta - \tilde{\eta})||_{-1,q}$$

$$+ 2\mu ||\text{div}[\sigma_\varepsilon(||D\xi||^2)D\xi - \sigma_\varepsilon(||D\tilde{\xi}||^2)D\tilde{\xi}]||_{-1,q} + ||(\zeta - \tilde{\zeta})f||_{-1,q}. $$  

(3.25)

We estimate each term on the right hand-side of (3.25) as follows:

$$||\text{div}(\tilde{\xi} \otimes \xi - \xi \otimes \tilde{\xi})||_{-1,q} \leq C||\tilde{\xi} \otimes \xi - \xi \otimes \tilde{\xi}||_q \leq 2CC_\mu(C_p^2 + 1)^{\frac{1}{4}} \rho_1 ||\nabla(\xi - \tilde{\xi})||_q;$$  

(3.26)

$$2\mu ||\text{div}[\sigma_\varepsilon(||D\xi||^2)D\xi - \sigma_\varepsilon(||D\tilde{\xi}||^2)D\tilde{\xi}]||_{-1,q} \leq C\mu||\sigma_\varepsilon(||D\xi||^2)D\xi - \sigma_\varepsilon(||D\tilde{\xi}||^2)D\tilde{\xi}||_q$$

$$\leq C\overline{\lambda}_p \ell(2\rho_1) ||\nabla(\xi - \tilde{\xi})||_q;$$  

(3.27)

$$2\nu_\varepsilon ||\text{rot}(\eta - \tilde{\eta})||_{-1,q} \leq C\nu_\varepsilon ||\eta - \tilde{\eta}||_q \leq C\nu_\varepsilon ||\eta - \tilde{\eta}||_{1,r} \leq C(C_p + 1)\nu_\varepsilon ||\nabla(\eta - \tilde{\eta})||_r;$$  

(3.28)

$$||\zeta - \tilde{\zeta}||f||_{-1,q} \leq C||\zeta - \tilde{\zeta}||\infty ||f||_q \leq C||\zeta - \tilde{\zeta}||\infty ||f||_q \leq CC_\overline{E}(C_p + 1) ||f||_q ||\nabla(\zeta - \tilde{\zeta})||_q. $$  

(3.29)

Inserting (3.26)–(3.29) into (3.25), we obtain

$$||\nabla(u_\varepsilon - \tilde{u}_\varepsilon)||_q \leq \frac{CC_{-1}}{\mu} \left[ 2C\mu(C_p^2 + 1)^{\frac{1}{4}} \rho_1 ||\nabla(\xi - \tilde{\xi})||_q + (C_p + 1)\nu_\varepsilon ||\nabla(\eta - \tilde{\eta})||_r \right]$$

$$+ C\overline{\lambda}_p \ell(2\rho_1) ||\nabla(\xi - \tilde{\xi})||_q + C_\overline{E}(C_p + 1) ||f||_q ||\nabla(\zeta - \tilde{\zeta})||_s$$

$$\leq \overline{\lambda}_4 \left[ \frac{1}{\mu} \rho_1 + \mu \nu_\varepsilon \overline{\lambda}_p \ell(2\rho_1) \max\{||\nabla(\xi - \tilde{\xi})||_q, ||\nabla(\eta - \tilde{\eta})||_r, ||\nabla(\zeta - \tilde{\zeta})||_s\} \right],$$

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where $\lambda_4 = CC_{-1}\max\{2C_p(C_p^p + 1)^{\frac{3}{2}}, C_p + 1, C_{-1}(C_p + 1)\}$.

On the other hand, by the theory of elliptic equation, there exists a positive constant $C_3$ such that

$$\|\nabla(\omega_\epsilon - \tilde{\omega}_\epsilon)\|_r \leq \|\nabla(\omega_\epsilon - \tilde{\omega}_\epsilon)\|_r$$

(3.31)

For each term on the right-hand side of (3.31), there have

$$\|2\nu_r \text{rot} (\xi - \tilde{\xi})\|_r \leq C\nu_r \|\nabla(\xi - \tilde{\xi})\|_q,$$

(3.32)

$$\|4\nu_r (\eta - \tilde{\eta})\|_r \leq 4\nu_r C_p \|\nabla(\eta - \tilde{\eta})\|_r,$$

(3.33)

$$\|(\xi - \tilde{\xi}) g\|_r \leq \|\xi - \tilde{\xi}\|_\infty \|g\|_r \leq C_{-1}(C_p + 1) \|\nabla(\xi - \tilde{\xi})\|_s \|g\|_r,$$

(3.34)

Combining (3.31) to (3.35), it follows that

$$\|\nabla(\omega_\epsilon - \tilde{\omega}_\epsilon)\|_r \leq \lambda_5(2\nu_r + 2\rho_1 + \|g\|_r)\max\{\|\nabla(\xi - \tilde{\xi})\|_q, \|\nabla(\eta - \tilde{\eta})\|_r, \|\nabla(\xi - \tilde{\xi})\|_s\},$$

(3.36)

where $\lambda_5 = C_{-1}\max\{4C_p, C, \frac{C_p}{C_{-1}}, C_p + 1, C_{-1}(C_p + 1)\}$.

Noticing that

$$-\kappa(\cdot, \theta_\epsilon) \Delta \theta_\epsilon + \kappa(\cdot, \tilde{\theta}_\epsilon) \Delta \tilde{\theta}_\epsilon = \kappa(\cdot, \tilde{\theta}_\epsilon) \Delta (\tilde{\theta}_\epsilon - \theta_\epsilon) + (\kappa(\cdot, \tilde{\theta}_\epsilon) - \kappa(\cdot, \theta_\epsilon)) \Delta \tilde{\theta}_\epsilon,$$

(3.37)

it follows from (3.24) that

$$\|\nabla(\theta_\epsilon - \tilde{\theta}_\epsilon)\|_s \leq \frac{1}{\kappa_1} \|H\|_s + \frac{1}{\kappa_1} \|\kappa(\cdot, \tilde{\theta}_\epsilon) \nabla \zeta_\epsilon^2 - \kappa(\cdot, \tilde{\theta}_\epsilon) \nabla \tilde{\zeta}_\epsilon^2\|_s + \frac{1}{\kappa_1} \|\kappa(\cdot, \tilde{\theta}_\epsilon) - \kappa(\cdot, \theta_\epsilon)\|_s \|\Delta \tilde{\theta}_\epsilon\|_s.$$

(3.38)

Recalling that $H = \Phi(\xi, \eta) - \Phi(\xi, \tilde{\eta}) - (\xi \cdot \nabla) \zeta + (\xi \cdot \nabla) \tilde{\zeta}$ and $\Phi(u, \omega) = \sum_{i=1}^4 \Phi_i$. In the sequel, we shall derive estimates for each term on the right-hand side of (3.38) one by one.

The first term can be estimated as follows.

$$\|\Phi_1(\xi, \eta) - \Phi_1(\xi, \tilde{\eta})\|_s \leq C \left( \int_\Omega \|\nabla(\xi - \tilde{\xi}) + (\tilde{\eta} - \eta)\| d\Omega \right)^{1/s}$$

$$\leq \sup_{x \in \Omega} \left( \|\nabla(\xi - \tilde{\xi}) + (\tilde{\eta} - \eta)\| + \|\nabla(\xi - \tilde{\xi}) + (\tilde{\eta} - \eta)\| \right)^{1/s}.$$
Combining the above estimates, taking $p_1 \in C(E(C_p + 1)) \|\nabla (\eta - \bar{\eta})\|_r$;
\[
\sum_{i=2}^{4} \|\Phi_1(\eta) - \Phi_2(\bar{\eta})\|_s \leq C p_1 \|\nabla (\eta - \bar{\eta})\|_s \leq C p_1 \|\nabla (\eta - \bar{\eta})\|_r;
\]
\[
\|\hat{\xi} \cdot \nabla \hat{\zeta} - (\xi \cdot \nabla \zeta)\|_s = \|\hat{\xi} - \xi\| \nabla \hat{\zeta} + \xi \nabla (\hat{\zeta} - \zeta)\|_s
\leq \|\hat{\xi} - \xi\| \nabla \hat{\zeta} + \xi \nabla (\hat{\zeta} - \zeta)\|_s
\leq C \|\hat{\xi} - \xi\|_{\infty} \nabla \hat{\zeta} + \xi \nabla (\hat{\zeta} - \zeta)\|_s
\leq C \frac{e_{C_p + 1}}{C_p} p_1 \|\nabla (\hat{\xi} - \xi)\|_q + (C_p + 1) p_1 \|\nabla (\hat{\zeta} - \zeta)\|_s;
\]
whence
\[
\|H\|_s \leq \overline{x}_6 (4 p_1) \max\{\|\nabla (\hat{\xi} - \xi)\|_q, \|\nabla (\hat{\eta} - \eta)\|_r, \|\nabla (\hat{\zeta} - \zeta)\|_s\},
\]
where $\overline{x}_6 = \max\{C, C[C + C\frac{e_{C_p + 1}}{C_p}], \frac{C \frac{e_{C_p + 1}}{C_p}}{C_p} + 1\}$.

For the second term
\[
\|\kappa' (\cdot, \zeta) \nabla \zeta^2 - \kappa' (\cdot, \hat{\zeta}) \nabla \hat{\zeta}^2\|_s
= \|\kappa' (\cdot, \zeta) - \kappa' (\cdot, \hat{\zeta})\| \nabla \zeta^2 + \kappa' (\cdot, \hat{\zeta}) (|\nabla \zeta^2| - |\nabla \hat{\zeta}^2|)\|_s
\leq \lambda \|\kappa' (\cdot, \zeta) - \kappa' (\cdot, \hat{\zeta})\|_{\infty} \|\nabla \zeta^2\|_s + \|\kappa' (\cdot, \hat{\zeta}) - \kappa' (\cdot, 0)|\|\nabla \zeta^2| - |\nabla \hat{\zeta}^2|)\|_s
\leq \lambda \|\kappa' (\cdot, \zeta) - \kappa' (\cdot, \hat{\zeta})\|_{\infty} \|\nabla \zeta^2\|_s + \lambda \|\kappa' (\cdot, \hat{\zeta}) - \kappa' (\cdot, 0)|\|\nabla \zeta^2| - |\nabla \hat{\zeta}^2|)\|_s
\leq \lambda C \frac{e_{C_p + 1}}{C_p} (C_p + 1) \|\nabla \zeta\|_{1,s} \|\nabla (\zeta - \hat{\zeta})\|_s + \lambda \|\nabla \zeta\|_{1,s} \|\nabla (\zeta - \hat{\zeta})\|_s
\leq \lambda \|\nabla \zeta\|_{1,s} \|\nabla (\zeta - \hat{\zeta})\|_s + 2 \lambda \|\nabla \zeta\|_{1,s} \|\nabla (\zeta - \hat{\zeta})\|_s;
\]
Finally, due to $|\kappa(\cdot, a) - \kappa(\cdot, b)| \leq \lambda (|a| + |b|)$, for $a, b \in \mathbb{R}$ and $\|\Delta \theta\|_s \leq \|\nabla \theta\|_1, \|\theta\|_s$, we have
\[
\|\kappa(\cdot, \bar{\theta}_e) - \kappa(\cdot, \theta_e)\| \Delta \theta_e\|_s \leq 2 \lambda \rho_3^2 (C_p + 1) \|\nabla (\theta - \hat{\theta}_e)\|_s.
\]
Combining (3.38)-(3.40)-(3.42), we arrive at
\[
(1 - 2 \frac{\lambda}{\kappa_1} \rho_1^2 (C_p + 1)^2) \|\nabla (\theta - \hat{\theta}_e)\|_s \leq 4 \overline{x}_6 \rho_1 \max\{\|\nabla (\hat{\xi} - \xi)\|_q, \|\nabla (\hat{\eta} - \eta)\|_r, \|\nabla (\hat{\zeta} - \zeta)\|_s\}
+ \frac{\lambda}{\kappa_1} (C_p + 1) \rho_1 \|\nabla (\zeta - \hat{\xi})\|_s + \frac{2 \lambda}{\kappa_1} \rho_1 (C_p + 1) \rho_1 \|\nabla (\zeta - \hat{\zeta})\|_s
\leq \rho_1 \overline{x}_7 \max\{\|\nabla (\hat{\xi} - \xi)\|_q, \|\nabla (\hat{\eta} - \eta)\|_r, \|\nabla (\hat{\zeta} - \zeta)\|_s\},
\]
where $\overline{x}_7 = \frac{3}{\kappa_1} \max\{4 \overline{x}_6, \lambda C (C_p + 1), 2 \lambda (C_p + 1)\}$.
Combining the above estimates, taking $p_1$ such that $\frac{2}{\kappa_1} \lambda \rho_3^2 (C_p + 1)^2 \leq \frac{1}{2}$, we conclude that
\[
\max\{\|\nabla (u_e - \hat{u}_e)\|_q, \|\nabla (\omega_e - \hat{\omega}_e)\|_r, \|\nabla (\theta_e - \hat{\theta}_e)\|_s\}
\leq \left[ \frac{\overline{x}_4 \rho_1}{\mu} + \frac{\overline{x}_4 \nu_r}{\mu} + \frac{\overline{x}_4 \overline{S}_p (2 p_1)}{\mu} + \frac{\overline{x}_4}{\nu_r} \|f\|_q + 2 \overline{x}_5 \rho_1 + 2 \overline{x}_5 \rho_1 + \overline{x}_5 \|g\|_r + 2 \overline{x}_7 \rho_1 \right]
\cdot \max\{\|\nabla (\hat{\xi} - \xi)\|_q, \|\nabla (\hat{\eta} - \eta)\|_r, \|\nabla (\hat{\zeta} - \zeta)\|_s\}.
\]

Choosing $\overline{x}_0 = \max\{\overline{x}_1, 2\overline{x}_5, \overline{x}_7\}$, noticing $\rho_1 \leq \frac{2\overline{x}_1(\|f\|^2 + \nu_r)}{\mu}$ and taking into account the function $\ell$ is nondecreasing, $\ell(4\rho_1) \leq 4^{(p-2,1)^{+}}\ell(y)$, we finally obtain

$$\max\left\{\left\|\nabla(u_\epsilon - \hat{u}_\epsilon)\right\|_q, \left\|\nabla(\omega_\epsilon - \hat{\omega}_\epsilon)\right\|_r, \left\|\nabla(\theta_\epsilon - \hat{\theta}_\epsilon)\right\|_s\right\}$$

$$\leq \overline{x}_0\left[\frac{1}{\mu} + \frac{\nu_r}{\mu} + 3_\rho(2\rho_1) + \frac{\|f\|^2}{\mu} + \|g\| + \nu_r + 2\rho_1\right]$$

$$\cdot \max\left\{\|\nabla(\hat{\xi} - \xi)\|_q, \|\nabla(\hat{\eta} - \eta)\|_r, \|\nabla(\hat{\zeta} - \zeta)\|_s\right\}$$

$$\leq \overline{x}_0\left[\left(\frac{1}{\mu} + 2\right)\frac{2\overline{x}_1(\|f\|^2 + \nu_r)}{\mu} + \left(\frac{1}{\mu} + 1\right)\nu_r + \frac{\|f\|^2}{\mu} + \|g\| + \overline{3}_p4^{(p-2,1)^{+}}\left(\frac{\overline{x}_1(\|f\|^2 + \nu_r)}{\mu}\right)\right]$$

$$\cdot \left(1 + \frac{\overline{x}_1(\|f\|^2 + \nu_r)}{\mu}\right)^{(p-3)^{+}} \cdot \max\left\{\|\nabla(\hat{\xi} - \xi)\|_q, \|\nabla(\hat{\eta} - \eta)\|_r, \|\nabla(\hat{\zeta} - \zeta)\|_s\right\}$$

$$\leq 4^{(p-2,1)^{+}}\overline{x}_0\left[\left(\frac{1}{\mu} + 2\right)\frac{2\overline{x}_1(\|f\|^2 + \nu_r)}{\mu} + \left(\frac{1}{\mu} + 1\right)\nu_r + \frac{\|f\|^2}{\mu} + \|g\| + \overline{3}_p\left(\frac{\overline{x}_1(\|f\|^2 + \nu_r)}{\mu}\right)\right]$$

$$\cdot \left(1 + \frac{\overline{x}_1(\|f\|^2 + \nu_r)}{\mu}\right)^{(p-3)^{+}} \cdot \max\left\{\|\nabla(\hat{\xi} - \xi)\|_q, \|\nabla(\hat{\eta} - \eta)\|_r, \|\nabla(\hat{\zeta} - \zeta)\|_s\right\}.$$

Considering the space $Y := W_0^{1,q}(\Omega) \times W_0^{1,r}(\Omega) \times W_0^{1,s}(\Omega)$ with norm $\max\{\|\nabla \cdot \|_q, \|\nabla \cdot \|_r, \|\nabla \cdot \|_s\}$, inequality (3.43) implies that

$$\|T_x(\hat{\xi}, \hat{\eta}, \hat{\zeta}) - T_x(\xi, \eta, \zeta)\|_Y \leq 4^{(p-2,1)^{+}}\overline{x}_0\left[\left(\frac{1}{\mu} + 2\right)\frac{2\overline{x}_1(\|f\|^2 + \nu_r)}{\mu} + \left(\frac{1}{\mu} + 1\right)\nu_r + \frac{\|f\|^2}{\mu} + \|g\| + \overline{3}_p\left(\frac{\overline{x}_1(\|f\|^2 + \nu_r)}{\mu}\right)\right]$$

$$\cdot \left(1 + \frac{\overline{x}_1(\|f\|^2 + \nu_r)}{\mu}\right)^{(p-3)^{+}} \cdot \|\hat{\xi}, \hat{\eta}, \hat{\zeta} - (\xi, \eta, \zeta)\|_Y.$$ From which and hypothesis (3.23), we obtain $T_x : B_\rho \to B_\rho$ is a contraction in $W_0^{1,q}(\Omega) \times W_0^{1,r}(\Omega) \times W_0^{1,s}(\Omega)$.

**Step 4: Proof of Theorem 3.1.**

We observe that for $p \leq 3$, $\gamma_p = \frac{1}{2} = \frac{1}{4(p-2,1)^{+}}$ and for $p > 3$, $\gamma_p > \frac{1}{4(p-2,1)^{+}}$. Thus, by taking $\lambda = (\overline{x}_0, \overline{x}_1)$ and because of (3.2) implies (3.5) and (3.23), taking $X = V_{2,q} \times (W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)) \times (W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega))$, $Y = W_0^{1,q}(\Omega) \times W_0^{1,r}(\Omega) \times W_0^{1,s}(\Omega)$, $B = B_{\rho_1}$, according to Lemma 2.3, we know that $T_x$ has a unique fixed point on $B_{\rho_1}$. This completes the proof of Theorem 3.1.

**4 Proof of Theorem 2.1.**

Notice that for each $\epsilon > 0$, $(u_\epsilon, \omega_\epsilon, \theta_\epsilon)$ satisfies the following weak formula

$$\int_\Omega \left(2\mu(1 + \sqrt{\epsilon^2 + \|Du_\epsilon\|^2})^{(p-2)}Du_\epsilon : D(\Psi)dx - \int_\Omega (u_\epsilon \otimes u_\epsilon) : D(\Psi)dx \right)$$

$$= 2\nu_\epsilon \int_\Omega \text{rot}\omega_\epsilon \cdot \Psi dx + \int_\Omega \theta_\epsilon f \cdot \Psi dx, \quad \forall \Psi \in V,$$
The regularity of $u$, $\psi$ and $\theta$ follows from (3.10), (3.11) and (3.16). Theorem 2.1 is proved.

5 Conclusions

In this paper, we proved the existence and uniqueness of strong solutions for a class of steady non-Newtonian micropolar fluid equations with heat convection. As far as we can see, the known results are all regarding the Newtonian case, related result for such a problem of non-Newtonian type has not been considered yet. The results in this paper are new and generalize many related problems in the literature.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.
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Conflict of interest

The authors declare there is no conflicts of interest.

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