Global dynamics of a predator-prey system with immigration in both species

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Abstract: The vast majority of species in nature live in ecosystems that are not isolated, and the same is true for predator-prey ecological systems. With this work we extend a predator-prey model by considering the inclusion of an immigration term in both species. From a biological point of view, that allows us to achieve a more realistic model. A system with Holling type I functional response is considered and we study its global dynamics, and that allows not only to determine the behavior in a region of the plane \(\mathbb{R}^2\), but also to control the orbits that go or come from the infinity. First we study the local dynamics of the system, analyzing the singular points and their stability, as well as the possible behavior of the limit cycles when they exists. By using the Poincaré compactification we get to determine the global dynamics, studying the global phase portraits in the positive quadrant of the Poincaré disk, the region where the system is of interest from a biological point of view.

Keywords: predator-prey, immigration, stability, global dynamics, phase portrait.

1. Introduction and statement of the main result

In the field of biosystems, dynamical systems and differential equations have been widely used from decades. There are numerous works where these areas come together to try to solve problems in ecology, biology or medicine, from the classic Lotka-Volterra or SIR models to some very recent works such as [17], where the authors study vegetation patterns with the aim of preserving or restoring them, by studying a vegetation model with non-local root interaction, or [15], where the authors relate human activities to the evolution of ecosystems, trying to provide solutions for biodiversity conservation.

One special area that has aroused the interest of researchers since the beginnings of Biomathematics is the one of predator-prey systems. Advances in these models are continually being made, some recent work can be found for example at [1, 5, 7, 16, 18, 20–22]. Although there are many contributions in the literature on the predator-prey models, due to the complexity of the real systems being represented, there are many features that still need to be studied in greater detail. For example,
to consider the effects of the presence of some number of immigrants, as most systems in the nature are not isolated, and in all major branches of the animal kingdom there are migrating species, from crustaceans or insects to big mammals [4]. This migratory phenomenon must be taken into account in models of very different scales, since it affects from small *zooplankton* species (1 mm in length) to large blue whales (up to 27 meters in length) [11, 14].

In recent years, some work has been done studying the effects of the presence of immigrants on different populations and species, and doing that under different hypothesis and with different mathematical tools and models. There are some works that use delay equations, because delayed migration can occur when the individuals encounter some barriers, see for example [2, 3, 6, 28]. Other recent works consider fractional order models, as in [27].

Among all this great variety of works, some have been selected, analyzed and compared in [8]. We highlight the work of J. Sugie [25], in which the authors include the effect of a constant immigration rate affecting the prey species in the classical Rosenzweig-MacArthur model. This allows them to state some ecological conclusions that point to the fact that inclusion of immigration allows for the coexistence of species, and that considering immigration can be important and have considerable effects even in simple models.

Another very recent work, in which immigration is considered in both the prey and the predator is the one of M. Priyanka et al. [23], but in that case immigration is not constant, but proportional to the population, and it is also considered combined with other characteristics such as harvesting, which is represented by a term with the same form to that of immigration.

The work of T. Tahara et al., [26], analyzes different predator-prey models with immigration. In addition to different types of functional responses, they consider two representations of immigration: one as a constant and another as a function of the type $c/x$. Several numerical simulations are carried out to show how the inclusion of immigration allows the stabilization of populations. Although some of the cases with immigration in only one of the species are briefly studied, no analytical study is made of the case with immigration in both species.

Other works deal with predator-prey models in higher dimensions. This is the case of the work of D. Mukherjee [19], in which it considered a predator prey model in dimension three in which there are a predator species, a prey species and a competitor of the prey. In this case they also considered immigration rate is constant, but is only take into account in the prey species. The author studies the existence and stability of the equilibria and shows that Hopf bifurcation can occur under certain conditions.

We also found some discrete-time systems in which this type of predator-prey ecosystems with immigration are studied. Also in these cases, they are limited to consider immigration in only one of the species. For example, in [12], the authors extend the study of the Holling type I discrete system considered in [24], by adding a constant prey immigration rate. They study the equilibria and their topological classification, showing the changes in the dynamics obtained by the inclusion of the immigration term. Bifurcation is also analyzed, and bifurcation diagrams together with phase portraits are included, but they are obtained by numerical simulations and no directly from the analytical results.

Motivated by these works in the literature, we want to carry out a complete study of the global dynamics of a predator-prey system with immigration in both species. We will take the immigration rate as a constant, as is done in many of the works mentioned above, being reasonable from a biological point of view. Then, we will be advancing in the study of the dynamics in two ways: on the one hand
by adding a new characteristic, immigration, and on the other hand by making a study of the dynamics that is not limited to local behavior but through a compactification we will be able to know the behavior at infinity. This has been done recently with other classical predator-prey models, as in [9].

We consider that this work can be very useful to study ecosystems in which both prey and predators present migratory behaviors, something that in fact occurs in the nature. The mathematical study of the system proposed below will allow one to predict and interpret the behavior of the species in ecosystems. We would like to point out that we have not found any other work in which immigration is considered for both species, and this is one of the innovative aspects of the work presented. On the other hand, in general, the study of the global dynamics of the systems is not carried out either, but this is something that allows to have a complete control of the dynamics, without limiting it to local behaviors, being able to work in a compact region and thus being possible to classify in its totality all the possible dynamic behaviors of a system.

Therefore, we hope that this model, as well as the classification of its global dynamics, serves for direct application to ecological problems, and also as a starting point for future models that consider immigration in both populations, the prey and the predator.

In general we can consider the system

\[
\dot{x} = rx - \frac{ax^{1+\alpha}y}{1 + hx^{1+\alpha}} + c_1, \\
\dot{y} = \frac{bx^{1+\alpha}y}{1 + hx^{1+\alpha}} - ny + c_2,
\]

(1.1)

where \(r\) represent the growth rate of prey; \(n\) is the death rate of predator; \(a\) is the rate of predation; \(b\) is the conversion rate of eaten prey into new predators; \(h\) and \(\alpha\) are the functional response coefficients (which involve for example the handling time); and \(c_1\) and \(c_2\) are the immigration rates of prey and predator, respectively. All these real parameters are positive due to their biological meaning.

To begin with, in the present work we deal with the case with \(\alpha = h = 0\), which corresponds with a Holling type I functional response. Our main result is the following:

**Theorem 1.1.** The global phase portrait of system (1.1) with \(\alpha = h = 0\), in the positive quadrant of the Poincaré disc is one of the following:

**Figure 1.** Global phase portraits of system (1.1) with \(\alpha = h = 0\) in the positive quadrant of the Poincaré disk.
The importance of this result is that it allows us to determine that for any value of the parameters, the behavior will follow one of these two schemes. Furthermore, we emphasize that the given phase portraits are in the positive quadrant of the Poincaré disk, i.e., they are global portraits, which makes possible to control the behavior near the infinity, and to determine how the orbits coming from or going to infinity behave.

From an ecological point of view, in the first phase portrait there is a singular point which represents the coexistence of both species, and we observe that regardless of the initial condition considered, that is, the initial number of prey and predators, the solution predicts an evolution to that singular point of coexistence. In the second phase portrait, there are two regions, the one inside the limit cycle and the one outside it. Within the limit cycle, the behavior is similar as that mentioned in the first phase portrait, since given any initial condition in that region, the number of prey and predators will tend to the value at the equilibrium point. For the initial conditions outside the limit cycle, the solutions increasingly approach the limit cycle on the outside, which means that, in practice, there will be an oscillation in the number of prey and predators, with a behavior of cyclic type.

2. Proof of the result

2.1. Finite singular points

System (1.1) with $h = \alpha = 0$ corresponds with a Holling type I functional response:

$$\begin{align*}
\dot{x} &= rx - axy + c_1, \\
\dot{y} &= bxy - ny + c_2,
\end{align*}$$

(2.1)

In order to make the reading more clear, let introduce the notation

$$R = \sqrt{4bc_1nr + (bc_1 + ac_2 - nr)^2}.$$

There are two general solutions for $\dot{x} = \dot{y} = 0$, with the expressions

$$\left(\frac{-bc_1 - ac_2 + nr + R}{2br}, \frac{bc_1 + ac_2 + nr + R}{2an}\right) \quad \text{and} \quad \left(\frac{-bc_1 - ac_2 + nr - R}{2br}, \frac{bc_1 + ac_2 + nr - R}{2an}\right),$$

which always are well defined as all the parameters are positive and $R$ is a real number for all the values of the parameters. In any case, given the motivation behind the formulation of the system, we are only interested in the non-negative singular points, so we analyze the sign and position of the points given by these expressions. The following results allow us to characterize the location of these singular points.

**Proposition 2.1.** The region $C = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$ is positively invariant.

**Proof.** The fact that the region $C$ is positively invariant can be deduced from the fact that

$$\dot{x} \big|_{x=0} = c_1 > 0, \quad \text{and} \quad \dot{y} \big|_{y=0} = c_2 > 0,$$

(2.2)
which means that the orbits of the system enter the region at any point of the boundary of $C$, when $t$ moves on in the positive sense.

\[ \square \]

**Corollary 2.2.** There are no singular points representing the survival of only one of the species.

**Proof.** As the positive axes $\{(0, y) \in \mathbb{R}^2 \mid y \geq 0\}$ and $\{(x, 0) \in \mathbb{R}^2 \mid x \geq 0\}$ are not invariant lines, and the flow is transversal to them, the singular points of system (2.1) can not be over the axes. \[ \square \]

**Theorem 2.3.** System (2.1) has exactly one positive singular point

\[ P = \left( \frac{-bc_1 - ac_2 + nr + R}{2br}, \frac{bc_1 + ac_2 + nr + R}{2an} \right), \]

for any values of the parameters.

**Proof.** Let us prove that this point is always positive. For the second component this is trivial as all the parameters are positive. For the first component, suppose that it is negative, it is, $-bc_1-ac_2+nr+R < 0$. Then

\[ R = \sqrt{4bc_1nr + (bc_1 + ac_2 - nr)^2} < bc_1 + ac_2 - nr \]

and squaring the expressions

\[ 4bc_1nr + (bc_1 + ac_2 - nr)^2 < (bc_1 + ac_2 - nr)^2 \Rightarrow 4bc_1nr < 0, \]

which contradicts the fact that all the parameters are positive. An analogous reasoning allows to prove that the other solution of $\dot{x} = \dot{y} = 0$ is never positive. If we suppose it is positive, then $-bc_1-ac_2+nr-R > 0$ and again squaring the expressions one can get

\[ (bc_1 + ac_2 - nr)^2 > 4bc_1nr + (bc_1 + ac_2 - nr)^2 \Rightarrow 4bc_1nr < 0, \]

which is a contradiction. Then the system has always exactly one positive singular point. \[ \square \]

**Proposition 2.4.** The singular point $P$ of system (2.1) is asymptotically stable and has the following phase portraits depending on the parameters:

1. It is a stable focus if $((n + r)(bc_1 + ac_2) + (n - r)(nr + R))^2 < 16n^2r^2R$.
2. It is a stable node if $((n + r)(bc_1 + ac_2) + (n - r)(nr + R))^2 > 16n^2r^2R$.

**Proof.** The Jacobian matrix of system (2.1) at the singular point $P$ is

\[ M = \begin{pmatrix} \frac{r - bc_1 - ac_2 + nr + R}{2n} & \frac{a(bc_1 + ac_2 - nr - R)}{2br} \\ \frac{b(bc_1 + ac_2 + nr + R)}{2an} & \frac{-bc_1 - ac_2 + nr - R}{2r} \end{pmatrix}, \]

and it has eigenvalues

\[ \lambda_1 = -\frac{1}{4nr}((ac_2 + bc_1)(n + r) + (nr - R)(n - r)) \]
\( \lambda_2 = \frac{1}{4nr} \left( (ac_2 + bc_1)(n + r) + (nr - R)(n - r) \right) - \sqrt{-16n^2r^2R + ((n + r)(bc_1 + ac_2) + (n - r)(nr - R))^2} \),

Then the trace is

\[
tr(M) = r - \frac{bc_1 + ac_2 + nr + R}{2n} - \frac{bc_1 + ac_2 + nr - R}{2r}
\]

and it can be proved that this expression is always negative. First

\[
r - \frac{bc_1 + ac_2 + nr + R}{2n} = - \frac{bc_1 + ac_2 - nr + R}{2n}
\]

is negative as

\[
bc_1 + ac_2 - nr + R = bc_1 + ac_2 - nr + \sqrt{(bc_1 + ac_2 - nr)^2 + 4bc_1nr} > 0.
\]

Furthermore, taking into account that

\[
4bc_1nr + (bc_1 + ac_2 - nr)^2 = 4bc_1nr + (bc_1 - nr)^2 + a^2c_2^2 + 2(bc_1 - nr)ac_2
\]

\[
= 4bc_1nr + b^2c_1^2 + n^2r^2 - 2bc_1nr + a^2c_2^2 + 2(bc_1 - nr)ac_2
\]

\[
= b^2c_1^2 + n^2r^2 + 2bc_1nr + a^2c_2^2 + 2(bc_1 - nr)ac_2
\]

\[
= (bc_1 + nr)^2 + a^2c_2^2 + 2(bc_1 + nr)ac_2 - 4nrac_2
\]

it holds that

\[
\frac{bc_1 + ac_2 + nr - \sqrt{(bc_1 + ac_2 - nr)^2 + 4bc_1nr}}{2r} = - \frac{bc_1 + ac_2 + nr - \sqrt{(bc_1 + ac_2 + nr)^2 - 4bc_1nr}}{2r}
\]

and this is negative as \(bc_1 + ac_2 + nr > \sqrt{(bc_1 + ac_2 + nr)^2 - 4nrac_2}\). Then we have that the trace is always negative for any values of the parameters.

The determinant is

\[
\lambda_1\lambda_2 = \sqrt{b^2c_1^2 + (ac_2 - nr)^2 + 2bc_1(ac_2 + nr)},
\]

which is positive.

If the radicand in the expressions of \(\lambda_1, \lambda_2\) is negative, i.e., if \(((n + r)(bc_1 + ac_2) + (n - r)(nr + R))^2 < 16n^2r^2R\), then the two eigenvalues are complex, and the singular point \(P\) is a stable focus as \(\lambda_1 + \lambda_2 = 2\text{Re}(\lambda_1) = 2\text{Re}(\lambda_2) < 0\). Then \(P\) is asymptotically stable.

If the radicand is positive, i.e., if \(((n + r)(bc_1 + ac_2) + (n - r)(nr + R))^2 > 16n^2r^2R\), the eigenvalues are real, and since \(\lambda_1, \lambda_2 > 0\), the singular point is a node, and taking into account that \(\lambda_1 + \lambda_2 < 0\), then both eigenvalues are negative and the node is stable, so \(P\) is asymptotically stable.

\(\square\)
2.2. Dynamics at the infinity

In order to study the global dynamics of the system, we will use the Poincaré compactification, as it makes possible to control the dynamics of a polynomial differential system near the infinity. The following is an introduction to the basic notions of this technique, but more details can be found in chapter 5 of [10].

Consider the sphere \( S^2 = \{ y \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1 \} \), which is called the Poincaré sphere. In general, a planar polynomial system of the form

\[
\begin{align*}
\dot{x}_1 &= P(x_1, x_2), \\
\dot{x}_2 &= Q(x_1, x_2),
\end{align*}
\]

can be projected into the sphere, obtaining an induced vector field in \( S^2 \setminus S^1 \). Note that we can identify \( \mathbb{R}^2 \) with the tangent plane to the sphere at the point \((0, 0, 1)\).

Then we can obtain the induced vector field by means of the central projections \( f^+ : \mathbb{R}^2 \to S^2 \) and \( f^- : \mathbb{R}^2 \to S^2 \), defined by

\[
\begin{align*}
f^+(x) &= \left( \frac{x_1}{\Delta(x)}, \frac{x_2}{\Delta(x)}, \frac{1}{\Delta(x)} \right) \quad \text{and} \quad f^-(x) = \left( \frac{-x_1}{\Delta(x)}, \frac{-x_2}{\Delta(x)}, \frac{-1}{\Delta(x)} \right),
\end{align*}
\]

where \( \Delta(x) = \sqrt{x_1^2 + x_2^2 + 1} \).

The differential \( Df^+ \) and \( Df^- \) provide a vector field in the northern and southern hemisphere, respectively, and we can extend analytically this vector field to the points of the equator multiplying the field by \( y_3^d \), where \( d \) is the degree of the original vector field in \( \mathbb{R}^2 \). This is important as the points of the equator \( S^1 \) of \( S^2 \) correspond with the points of infinity of \( \mathbb{R}^2 \).

To make calculations, we work in the local charts \((U_i, \phi_i)\) and \((V_i, \psi_i)\) of the sphere \( S^2 \), where \( U_i = \{ y \in S^2 : y_i > 0 \} \), \( V_i = \{ y \in S^2 : y_i < 0 \} \), \( \phi_i : U_i \to \mathbb{R}^2 \) and \( \psi_i : V_i \to \mathbb{R}^2 \) for \( i = 1, 2, 3 \) with \( \phi_i(y) = \psi_i(y) = (y_m/y_i, y_n/y_i) \) for \( m < n \) and \( m, n \neq i \).

The extended field is called the Poincaré compactification of the original vector field. The expression of the Poincaré compactification in the local chart \((U_1, \phi_1)\) is

\[
\dot{u} = v^d \left[ -u P \left( \frac{1}{v}, \frac{u}{v} \right) + Q \left( \frac{1}{v}, \frac{u}{v} \right) \right], \quad \dot{v} = -v^{d+1} P \left( \frac{1}{v}, \frac{u}{v} \right), \quad (2.3)
\]

in the local chart \((U_2, \phi_2)\) is

\[
\dot{u} = v^d \left[ P \left( \frac{u}{v}, \frac{1}{v} \right) - u Q \left( \frac{u}{v}, \frac{1}{v} \right) \right], \quad \dot{v} = -v^{d+1} Q \left( \frac{u}{v}, \frac{1}{v} \right), \quad (2.4)
\]

and in the local chart \((U_3, \phi_3)\) the expression is

\[
\dot{u} = P(u, v), \quad \dot{v} = Q(u, v). \quad (2.5)
\]

In the charts \((V_i, \psi_i)\), with \( i = 1, 2, 3 \), the system is the same as in the charts \((U_i, \phi_i)\) multiplied by \((-1)^{d-1}\).

As we want to study the behavior near the infinity, we must study the infinite singular points, i.e., those which lie on the equator of the sphere.
It is sufficient to study the infinite points on the local chart $U_1$ and the origin of the local chart $U_2$, because if $y \in S^1$ is an infinite singular point, then $-y$ is also an infinite singular point and they have the same or opposite stability depending on whether the system has odd or even degree. In the case of this work, since our system is motivated by a real population problem, it will only be of interest to study it for positive variables.

Then, we shall present the phase portraits of the polynomial differential system in the positive quadrant of the Poincaré disc, which is the orthogonal projection of the northern hemisphere of $S^2$ onto the plane $y_3 = 0$.

According to (2.3), in chart $U_1$ system (2.1) has the expression

$$\dot{u} = -c_1uv^2 + au^2 - (n + r)uv + c_2v^2 + bu,$$
$$\dot{v} = -c_1v^3 + auv - rv^2.$$

The singular points over $v = 0$ are the origin of $U_1$ and the point $(-b/a, 0)$. As this second point is not on the positive quadrant of the Poincaré disk, it is not relevant in the problem we are studying. The linear part of system (2.6) at the origin is the matrix

$$L = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix},$$

so we have a semi-hyperbolic singular point. In order to determine its phase portrait Theorem 2.19 in [10] can be used. As the singular point is the origin of the system and the Jacobian matrix is in the real normal form, we just compute the functions $f(v)$ and $g(v)$ in the theorem, obtaining

$$g(v) = -\frac{c_2}{b}v^2 + o(x^2),$$

so according to the theorem it turns out that $m = 2$ and $a_m = -c_2/b \neq 0$, and then the singular point is a saddle-node. Thus we know that the sectorial decomposition of the singular point has two hyperbolic sectors and one parabolic sector, but it is necessary to determine the orientation and position of both the three sectors.

Studying the flow over the horizontal axes we have that $\dot{v} \mid v = 0 = 0$ and $\dot{u} \mid v = 0 = au^2 + bu$. Then, in a neighborhood of the origin there are four possibilities for the position and orientation of the different sectors, which are given in Figure 2.

Figure 2. Different configurations for a saddle-node in the origin of system (2.6) and (2.7).
Studying the flow over the vertical axis, we have that \( \dot{u} \big|_{u=0} = c_2 v^2 > 0 \) and \( \dot{v} \big|_{u=0} = -v^2(c_1 v + r) \). Note that this is not possible in the configurations (a)-(c) in Figure 2, so the only possible phase portrait for the origin of the chart \( U_1 \) is the one given in Figure 2 (d).

According to (2.4), in chart \( U_2 \) system (2.1) writes
\[
\begin{align*}
\dot{u} &= -c_2 u v^2 - b u^2 + (n + r)uv + c_1 v^2 - au, \\
\dot{v} &= -c_2 v^3 - buv + nv^2.
\end{align*}
\] (2.7)

The origin of this chart, which is the only point at the infinity that we have to consider here, is a singular point. The Jacobian matrix of the system at the origin is
\[
J = \begin{pmatrix}
-a & 0 \\
0 & 0
\end{pmatrix},
\]
and again there is a semi-hyperbolic singular point. Applying Theorem 2.19 in [10], in this case the function
\[
g(v) = c_1/a v^2 + o(x^2),
\]
is obtained, so according to the theorem we have a saddle-node as \( m = 2 \) and \( a_m = c_1/a \neq 0 \).

2.3. A note on the existence of a limit cycle

The analytical proof of the existence of the limit cycle has turned out to be complicated. We have tried to apply Theorem 3.3. in [13], but although the bifurcation threshold with respect to certain parameters is easy to obtain, and the first genericity condition has been proved in some cases, we have get after tedious computations a first Lyapunov coefficient equal to zero. The computations for the second Lyapunov coefficient were not manageable. Then we have carried out some numerical simulations that point to the fact that the limit cycle could appear under certain conditions.

In Figure 3, we represent the solutions of system (1.1), with parameters \( r = a = b = n = 0.2, c_1 = 0.3, \) and \( c_2 = 0.18 \). In this case, from an ecological point of view, the two species, after an increase and decrease in the number of individuals, stabilize in the population corresponding to the equilibrium point, and that number of prey and predator will be then constant in the future time. In other words, the solutions tend to the equilibrium point of the coexistence.

In Figure 4, the values of the parameters are \( r = 50, a = 2553/524288, c_1 = 5, b = 5/2147483, n = 1000/1024 \) and \( c_2 = 1 \), and we observe an oscillatory behavior that could represent that of an orbit approaching the limit cycle. From an ecological point of view, this situation represents an oscillatory behavior where the number of prey and predators increase and decrease periodically.
Figure 3. Solutions of system (1.1) with parameters $r = 50$, $a = 2553/524288$, $c_1 = 5$, $b = 5/2147483$, $n = 1000/1024$ and $c_2 = 1$ and initial conditions $x(0) = y(0) = 0.1$.

Figure 4. Solutions of system (1.1) with parameters $r = a = b = n = 0.2$, $c_1 = 0.3$, and $c_2 = 0.18$ and initial conditions $x(0) = y(0) = 0.1$.

2.4. Global phase portrait

To determine the global phase portrait on the positive quadrant of the Poincaré disk, we have to gather the local information obtained.

From Theorem 2.3 and 2.4 we now that there exists always one positive singular point, and it is a stable node or a stable focus, which is topologically equivalent.

At the infinity there are two singular points, the origins of the charts $U_1$ and $U_2$. The origin of $U_1$ has the phase portrait in Figure 2(d), and as the region of interest in our model is the positive quadrant, what we have is that there are orbits that enter into this quadrant from each point of the axis $u = 0$, except from the origin.

The origin of $U_2$ has the phase portrait in Figure 2(a). Again, focusing on the positive quadrant, we have that there exist a separatrix which leaves from this infinity singular point and gives rise to two different sector, one between the separatrix and the infinity and the other between the separatrix and the $u = 0$ axis. In the first one there are orbits which leave all from the origin of $U_2$, and in the second one there are orbits that enter the positive quadrant of the Poincaré disk from each one of the points of the $u = 0$ axis.

Accordingly to this, two global phase portraits can be obtained, those given in Figure 1, depending on whether or not there is a limit cycle surrounding the positive singular point.
In case there is such a limit cycle, the separatrix, as well as all the orbits entering the positive quadrant from each of the points on the $u = 0$ and $v = 0$ axes, go to the limit cycle when $t$ tends to infinity. In case it exists, the limit cycle is an attractor on the outside and a repulsor on the inside, and all orbits starting from some point in the region enclosed by the limit cycle, go to the positive singular point when $t$ tends to infinity. If there is no limit cycle, all orbits go to the positive singular point, which will then be globally asymptotically stable (in the positive quadrant of the Poincaré disk).

3. Discussion and conclusions

In this work we have studied a predator-prey system with Holling type I functional response, but with the particularity that we have added constant terms representing immigration in both the predator and the prey species. The main result of our work is that, in a predator-prey ecosystem with this characteristics, two dynamical behaviors can appear, which are given in Figure 1.

We highlight an important outcome: in both cases there is an asymptotically stable singular point, which from a biological point of view means that both populations can coexist. In the first case, the singular point is globally asymptotically stable, so the result is even stronger: regardless of the initial number of prey and predators, both species tend to the coexistence. If there exist a limit cycle, then it delimits the basins of attraction of the asymptotically stable singular point, and then, if the initial condition of the number of prey and predator is inside the region delimited by the limit cycle, then the populations also tend to the coexistence. In other case, for the rest of initial conditions, the trajectories tend to the limit cycle, which is attractor in the outside, and so the populations tend to a cyclical behavior. This result are topologically represented in Figure 1.

It has not been possible to analytically prove the existence of the limit cycle, and that is why we have relied on some numerical simulations, which allow us to see, for example in Figure 4, a certain oscillatory behavior, in which the number of prey and predators continually increases and decreases over time. Note that this does not allow us to conclude the existence of the limit cycle definitively.

This two dynamics obtained differ from the results in the model without immigration, in which there is a singular point which is a center, and all the other orbits are periodic orbits. Then, from our study it can be concluded that the presence of a certain number of immigrants in both species, something that is not strange in the nature, can affect the dynamics of the ecosystem, and can lead to asymptotically stable coexistence equilibria.

These results differ from those obtained in the classical Lotka-Volterra model in which there is a stable singular point that is topologically a center and all other solutions are periodic orbits. The results point in the same direction as previous work in which immigration was considered only in one of the species as in [25] or [26]. It is difficult to compare with some other works where we do not see clearly which is the effect of immigration as it appears combined with different types of properties of the ecological systems, as it can be the case of [23]. We have not found works in which immigration in both species is considered and neither works in which the global dynamics is studied, so we can compare with models in conditions similar to ours.

As open problems for the future, it will be interesting to find a specific set of parameters for which the limit cycle exists, and also replace the constant immigration rate by other functions that can also represent the number of immigrants in both species.
Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

Appendix

As stated in subsection 2.3, trying to prove the existence of a limit cycle that appears by Hopf bifurcation, we have computed the first Lyapunov coefficient, although it has turned out to be zero, so it was not possible to conclude the existence of the bifurcation. Here we summarize the computations.

The Jacobian matrix at the equilibrium $P$ is

$$A(a) = \begin{pmatrix} r - \frac{bc_1 + ac_2 + nr + R}{2n} & a (bc_1 + ac_2 - nr - R) \\ \frac{b (bc_1 + ac_2 + nr + R)}{2an} & -\frac{bc_1 + ac_2 + nr - R}{2r} \end{pmatrix},$$

and it has eigenvalues $\mu(a) \pm \omega(a)i$, where

$$\mu(a) = -\frac{(ac_2 + bc_1)(n + r) + (nr - R)(n - r)}{4nr} \quad \text{and}$$

$$\omega(a) = \frac{1}{4nr} \sqrt{16n^2r^2R - ((n + r)(bc_1 + ac_2) + (n - r)(nr - R))^2}.$$  

We get $\mu(a_0) = 0$ for

$$a_0 = \frac{(r - n)(n + \sqrt{4bc_1 + n^2}) - 2bc_1}{2c_2}. \quad (3.2)$$

Then if the condition $16n^2r^2R - ((n + r)(bc_1 + ac_2) + (n - r)(nr - R))^2 > 0$ holds, at $a = a_0$ the equilibrium point $P$ has a pair of pure imaginary eigenvalues $\pm i\omega(a)$ and the system will have a Hopf bifurcation if some Lyapunov constant is nonzero and $(d\mu/da)(a_0) \neq 0$. Note that we would be interested in this case when also the expression for $a_0$ is positive, but this is not a problem.

It is necessary to check if the genericity conditions are satisfied (see [13, Theorem 3.3]). We check
that the transversality condition is satisfied as

\[
\frac{d\mu}{da}(a_0) = \frac{c_2(n-r)(bc_1 - nr)}{4nr \sqrt{\frac{1}{4} \left( n \left( \sqrt{4bc_1 + n^2 + r} \right) - r \sqrt{4bc_1 + n^2 + n^2} \right)^2 + 4bc_1nr}} - \frac{c_2(n-r) \left( \left( \frac{1}{2}(n-r) \left( \sqrt{4bc_1 + n^2 + n} \right) + nr \right)^2 + 4bc_1nr + nr \right)}{8nr \sqrt{\frac{1}{4} \left( n \left( \sqrt{4bc_1 + n^2 + r} \right) - r \sqrt{4bc_1 + n^2 + n^2} \right)^2 + 4bc_1nr}}
\]

(3.3)

and with the help of the software Mathematica v. 13.3.1 we have proved that this expression is always negative.

To check the second condition the first Lyapunov constant must be calculated. We fix the value \( a = a_0 \) and then the equilibrium \( P \) has the expression \( P_2 = (P_{21}, P_{22}) \) with

\[
P_{21} = \frac{\frac{1}{2}(n-r) \left( \sqrt{4bc_1 + n^2 + n} \right) + \sqrt{\frac{1}{2}(n-r) \left( \sqrt{4bc_1 + n^2 + n} + nr \right)^2 + 4bc_1nr + nr}}{2br},
\]

\[
P_{22} = -\frac{c_2 \left( \frac{1}{2}(n-r) \left( \sqrt{4bc_1 + n^2 + n} \right) + \sqrt{\frac{1}{2}(n-r) \left( \sqrt{4bc_1 + n^2 + n} + nr \right)^2 + 4bc_1nr + nr} \right)}{n \left( n-r \left( \sqrt{4bc_1 + n^2 + n} + 2bc_1 \right) \right)}
\]

(3.4)

We translate \( P \) to the origin of coordinates obtaining a system which can be represented as

\[
\dot{\varepsilon} = A\varepsilon + \frac{1}{2} B(\varepsilon, \varepsilon) + \frac{1}{6} C(\varepsilon, \varepsilon, \varepsilon),
\]

(3.5)

where \( A = A(a_0) \) and the multilinear functions \( B \) and \( C \) are given by

\[
B(\varepsilon, \eta) = \begin{pmatrix}
\frac{2bc_1 + (n-r) \left( n + \sqrt{4bc_1 + n^2} \right)}{2c_2} & 0 \\
0 & \frac{2bc_1 + (n-r) \left( n + \sqrt{4bc_1 + n^2} \right)}{2c_2}
\end{pmatrix}
\]

\[
C(\varepsilon, \eta, \zeta) = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

We need to find two eigenvectors \( p, q \) of the matrix \( A \) verifying

\[
Aq = i\omega q, \quad A^T p = -i\omega p, \quad \text{and} \quad < p, q > = 1,
\]
Now we compute ω as for example \( q = (q_1, q_2)^T \) and \( p = (p_1, p_2)^T \) where

\[
q_1 = \frac{1}{8bc_1r} \left( (n-r) \left( \sqrt{4bc_1 + n^2 + n} \right) \sqrt{2n \left( -r^2 \sqrt{4bc_1 + n^2 + n} + 4bc_1(n+r)^2 + 2n^3 + 2n^2 \right)} + 
\right.
\]

\[
+ 2bc_1 \left( n \left( \sqrt{4bc_1 + n^2 - 3r} - r \sqrt{4bc_1 + n^2} + \sqrt{2n \left( -r^2 \sqrt{4bc_1 + n^2 + n} + 4bc_1(n+r)^2 + 3n^2 + 2r^2 \right)} \right) \right) \right)
\]

\[
q_2 = \frac{n \left( \sqrt{4bc_1 + n^2 + r} - r \sqrt{4bc_1 + n^2} - \sqrt{2n \left( -r^2 \sqrt{4bc_1 + n^2 + n} + 4bc_1(n+r)^2 + n^2 \right)} \right)}{4n} - i\omega,
\]

\[
p_1 = \frac{1}{D} \left( \frac{bc_1 \left( n \left( \sqrt{4bc_1 + n^2 - 3r} - r \sqrt{4bc_1 + n^2} - \sqrt{2n \left( -r^2 \sqrt{4bc_1 + n^2 + n} + 4bc_1(n+r)^2 + 3n^2 + 2r^2 \right)} \right) \right)}{2n((n-r) \left( \sqrt{4bc_1 + n^2 + n} + 2bc_1 \right)} \right),
\]

\[
p_2 = - \frac{n \left( \sqrt{4bc_1 + n^2 + r} - r \sqrt{4bc_1 + n^2} - \sqrt{2n \left( -r^2 \sqrt{4bc_1 + n^2 + n} + 4bc_1(n+r)^2 + n^2 \right)} \right)}{4nD} - i\frac{\omega}{D},
\]

where \( \omega = \omega(a_0) \) and

\[
D = \frac{1}{4} \left( r \sqrt{4bc_1 + n^2} - n \sqrt{4bc_1 + n^2} - \sqrt{2n \left( -r^2 \sqrt{4bc_1 + n^2 + n} + 4bc_1(n+r)^2 + 2n^3 + 2n^2 \right)} \right) + \omega^2.
\]

Now we compute

\[
g_{20} = \langle p, B(q, q) \rangle = \frac{g_{20}^1}{g_{20}^2},
\]

\[
g_{11} = \langle p, B(q, q) \rangle = -\frac{g_{11}^1}{g_{11}^2},
\]

\[
g_{21} = \langle p, C(q, q, \bar{q}) \rangle = 0,
\]
where

\[ g_{10}^{2} = nr(2bc_{1} + (n - r)(n + T)) \left( 2bc_{1} \left( 3n^{2} + n(T - 3r) + 2r^{2} - rT + S \right) + (n - r) \left( 2n^{2} + S \right) (n + T) \right) \]

\[ + nr^{2} \left( -T \sqrt{4bc_{1}(n + r)^{2} + 2n(n^{3} + n^{2}T + nr^{2} - r^{2}T)} + 2r^{3} + 3rS \right) + 3n^{3} \left( 16r^{2} - 2rT + S \right) \]

\[ + r^{3} \left( T \sqrt{4bc_{1}(n + r)^{2} + 2n(n^{3} + n^{2}T + nr^{2} - r^{2}T)} - 2r^{2}T - 2rS \right) + n^{2}r \left( -T \sqrt{4bc_{1}(n + r)^{2} + 2n(n^{3} + n^{2}T + nr^{2} - r^{2}T)} \right) \]

\[ + 8r^{3} - 16r^{3}T - 5rS \right) + n^{3} \left( T \sqrt{4bc_{1}(n + r)^{2} + 2n(n^{3} + n^{2}T + nr^{2} - r^{2}T)} - 32r^{3} + 20(T - 7r) \right) \]

\[ g_{11}^{2} = abc_{1}nr^{2} \left( 4 \left( n^{3} + n(T - 3r) - rT - S \right) \left( 2b_{1}c_{1}^{2} + 2bc_{1}(n - r)(2n - r + T) + (n - r)^{2}(n + T) \right) \right) \]

\[ + (2bc_{1} + (n - r)(n + T)) \left( -2nr^{2} - 2r(n - r) + 2T + 2rT + rS \right) \]

\[ + 2n \left( 16r^{2}r^{2} \sqrt{4bc_{1}nr + \frac{1}{4} (n^{2} + n(r + T) - rT)^{2} - \frac{1}{4} (n - r)^{2} (n^{2} + T(n + r) - nr + S)^{2}} \right) \]

\[ g_{11}^{2} = c_{2} \left( nr \left( n^{3} + n(T - 3r) - rT - S \right) \left( 2bc_{1} \left( 3n^{2} + n(T - 3r) + 2r^{2} - rT + S \right) + (n - r) \left( 2n^{2} + S \right) (n + T) \right) \right) \]

\[ + (2bc_{1} + (n - r)(n + T)) \left( 16n^{2}r^{2} \sqrt{4bc_{1}nr + \frac{1}{4} (n^{2} + n(r + T) - rT)^{2} - \frac{1}{4} (n - r)^{2} (n^{2} + T(n + r) - nr + S)^{2}} \right) \]

and

\[ T = \sqrt{4bc_{1} + n^{2}}, \]

\[ S = \sqrt{4bc_{1}(n + r)^{2} + 2n(n^{3} + n^{2}T + r^{2}(n - T))}. \]

The first Lyapunov coefficient is

\[ \ell_{1} = \frac{1}{2\omega^{2}} Re(ig_{20}g_{11} + \omega g_{21}) = 0. \]

References


