Optimal Investment Consumption Choices under Mispricing and Habit Formation

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Abstract: This paper studies the optimal consumption and investment for an agent, considering statistical arbitrary opportunities caused by mispriced stocks. The agent exhibits consumption habit formation and has access to a risk-free asset, a market index, and a pair of mispriced stocks. The optimization problem is to find the optimal consumption and investment strategies to maximize the expected utility from consumption and terminal wealth. The utility of consumption stems from the difference between the consumption and habit level. Based on the dynamic programming method, a verification theorem is provided and the analytical solution of the optimization problem is obtained. Numerical results show the behaviors of our formulas and make practical recommendations. By studying the sensitivity of consumption and investment strategies to habit formation, mispricing, and delta-neutral arbitrage strategy, we uncover and analyze the behaviors of the agent. Meanwhile, we define and discuss the wealth-equivalent utility loss under three cases, including ignoring habit formation, ignoring mispricing, and adopting the delta-neutral arbitrage strategy.

Keywords: Habit formation; Mispricing; Optimal strategies; Utility loss; Delta neutral arbitrage strategy

1. Introduction

The research on investment and consumption has always been a hot topic in the field of financial mathematics. Since Merton [17] first sets up and solves an investment consumption problem in the context of continuous-time, there has been a lot of relevant literature, such as Cocco et al. [5], Bilsen and Laeven [2], Lichtenstern et al. [13], Ma and Li [16], Bellalah et al. [1], and so on. These works all study the optimal asset allocation in the context of the financial market involving one or multi-risk
assets and explore the effect of market features on the optimal strategies, where market features include the stochastic volatility risk of the risky assets, inflation risk, and stochastic rate risk. However, in the real financial market, there exists a so-called mispricing phenomenon.

Mispricing is a price difference between a pair of assets, where two assets with identical or nearly identical contingent claim values should have the same or close to the same price during the same trading period, seeing Gu et al. [8]. The main reason for this kind of phenomenon is frictions caused by the immaturity of the financial market. Usually, such assets do not have the same or nearly the same price in different financial market. For example, there exists mispricing in some Chinese-company stocks (such as Bank of China, Agriculture Bank of China, and others) traded on both Chinese stock exchanges as share A and Hong Kong stock exchanges as shares H. Gu et al. [8] point out that since 2015, the Chinese government opened up simultaneous investment in Hong Kong and mainland China financial markets, which means that a mainland China investor is allowed to invest in designated Hong Kong stocks and vice versa. Therefore it is of timely significance to Chinese investors. If an investor takes the pairs trading strategy to the mispricing stock pairs, the market systemic risk will be hedged to some extent. Liu and Longstaff [14] obtain the optimal long-short investment strategy, which is an arbitrage opportunity caused by mispricing assets. After that, the optimal asset allocation problem with mispricing has followed with interest. Liu and Timmermann [15] show that conventional long-short delta neutral strategies are generally suboptimal and it can be optimal to simultaneously go long (or short) in two mispriced assets. Under a utility function framework, based on [15], Yi et al. [20] find that the mispricing has effect on the optimal portfolio selection. Gu et al. [8] show that the mispricing feature causes statistical arbitrage opportunities, which are particularly timely in the investment environment for markets in mainland China and Hong Kong. Gu et al. [9] disclose that liquidity has an important role in the so-called long-short (L-S) strategy, which corresponds statistical arbitrage afforded by mispricing. To the best of our knowledge, the existing literature involving investment and consumption does not consider mispricing, thus we introduce it in this paper.

In addition to the mispricing phenomenon, the formation of consumption habit is attracting the attention of more scholars. Constantinides [6] uncovers that models with habit formation can obtain a high equity premium with low-risk aversion and thus used habit formation to explain the equity premium puzzle. Afterwards, in the context of time separable power utility further, Browning and Collado [4] provide empirical support for this result. Munk [18] suggests that the utility associated with the consumption choice should depend on past consumption choices. Li et al. [12] show that it is unrealistic if one assumes that current satisfaction only relies on current consumption. Shi et al. [19] show that the existence of habit formation affects the optimal decision of the agent. Thus, in a more reasonable preference expression, one should consider the formation of consumption habit.

However, the above work either considers mispricing without considering the formation of agent’ consumption habit, in the context of investment portfolio framework, such as [8, 14, 15, 20, 9]; or only considers the formation of consumption habits without considering mispricing phenomena, such as [6, 4, 18, 12, 19]. To the best of our knowledge, we are the first to simultaneously consider mispricing phenomenon and habit formation in a consumption-investment problem. This is the first contributions. The second contribution is that we give a verification theorem and obtain closed-form expressions of optimal consumption investment strategies and the optimal value function for the agent who ignores mispricing and for the agent who takes the delta neutral arbitrage strategy for granted.

The third contribution of this paper is that by numerical analysis, we show the effects of mispricing
and habit formation on the optimal strategy. Specifically, the paper considers six cases of the model. That is, the case involving mispricing and habit formation, the case involving delta neutral strategy and habit formation, the case involving mispricing and no habit formation, the case involving delta neutral strategy and no habit formation, the case without mispricing but involving habit formation, and the case without both mispricing and habit formation. Detailed results are also provided. (i) For a longer time horizon, under considering mispricing or delta neutral strategy, habit formation damps consumption. However, habit formation raises consumption when ignoring mispricing or taking the delta-neutral strategy diminishes consumption. For a shorter time horizon, consumption increases with time \( t \) and the effect of habit formation and mispricing on consumption gradually disappears. (ii) Habit formation decreases the portfolio weights of wealth invested in the market index, while ignoring mispricing or adopting delta-neutral strategy will raise it. Meanwhile, the results show that there is almost no difference between ignoring mispricing and adopting delta-neutral strategy when investing in the market index. (iii) As the time horizon shortens, the portfolio weights of wealth invested in the pair of mispriced stocks decrease. (iv) We respectively define and analyze the wealth-equivalent utility loss under three cases, including ignoring habit formation, ignoring mispricing, and adopting the delta-neutral arbitrage strategy. Under considering mispricing, habit formation damps consumption, leading to the agent’s utility (from consumption) decreasing, and thus if one ignores habit formation, the utility will increase. In addition, ignoring mispricing or adopting the delta-neutral arbitrage strategy will lead to utility loss.

The remainder of this study is organized as follows: Section 2 introduces some necessary assumptions and formulates the model. Section 3 solves the optimal consumption and investment problem with mispricing and consumption habit. Section 4 shows numerical results to analyze the sensitivity of consumption and investment strategies to some parameters. Section 5 concludes this paper.

2. The model formulation

This section sets up a continuous-time investment and consumption model. Assume that an agent can trade in the financial market with no transaction costs or taxes. Let \( T > 0 \) be fixed constant as the time horizon and \( (\Omega, \mathcal{F}, \{\mathcal{F}(t)\}, P) \) be a complete probability space. \( \{\mathcal{F}(t)\} \) is the natural filtration and satisfies the usual conditions (i.e., \( \{\mathcal{F}(t)\} \) is \( P \)-complete and right-continuous). \( \mathcal{F}(t) \) represents the information available until time \( t \). All the stochastic processes and random variables introduced below are supposed to be well-defined and adapted to \( \{\mathcal{F}(t)\} \).

2.1. The financial market

Assume that the financial market in our model consists of one risk-free asset, one market index and a pair of stocks with mispricing. The price process of the risk-free asset, denoted by \( S_0(t) \), satisfies the following ordinary differential equation

\[
\frac{dS_0(t)}{S_0(t)} = rd(t),
\]

where \( r > 0 \) is the risk-free interest rate. The price process of the market index, \( S_m(t) \), reflects the market performance and follows a geometric Brownian motion

\[
\frac{dS_m(t)}{S_m(t)} = (r + \mu_m)dt + \sigma_m dB(t),
\]
where $\mu_m$ is the market risk premium, $\sigma_m > 0$ is the market volatility, and $\{B(t)\}$ is a standard Brownian motion. Let $S_1(t)$ and $S_2(t)$ denote mispriced price process of the pair of stocks and satisfy the following stochastic differential equations

\[
\frac{dS_1(t)}{S_1(t)} = (r + \mu_m)b \, dt + \beta \sigma_m dB(t) + \sigma dZ_1(t) + b dZ_1(t) - l_1 X(t) \, dt,
\]

\[\frac{dS_2(t)}{S_2(t)} = (r + \mu_m)b \, dt + \beta \sigma_m dB(t) + \sigma dZ_2(t) + b dZ_2(t) + l_2 X(t) \, dt,
\]

where $l_1, l_2, \sigma, b, \beta$ are constants. The term $\beta \sigma_m dB(t)$ is the systematic risk of the market, $\sigma dZ(t) + bdZ_i(t)$ is the idiosyncratic risk of the stock, $\sigma dZ(t)$ describes the common risk, and $bdZ_i(t)$ is the individual risk. $Z(t), Z_1(t), Z_2(t)$ are mutually independent Brownian motions and all independent of $B(t)$. $X(t) = \ln S_1(t) - \ln S_2(t)$ denotes the pricing error or mispricing between two stocks. Then $l_i X(t)$ describes the effect of mispricing on the $i$th stock. Meanwhile, by the Itô formula, the dynamics of the pricing error $X(t)$ can be given

\[dX(t) = (l_1 + l_2)(0 - X(t)) \, dt + bdZ_1(t) - bdZ_2(t), X(0) = x_0.\]  

It is easy to see that $X(t)$ is a Gaussian mean reverting process; the long-run mean of $X(t)$ is 0 and the mean reversion rate of it is $l_1 + l_2$. In Eq. (2.4), $l_1$ and $l_2$ can not equal zero at the same time, otherwise, there is no pricing error. Assume $l_1 + l_2 > 0$ to assure that $X(t)$ is stationary (cf. [15],[8] and [20]). Sometimes, we can view $l_1$ and $l_2$ as liquidities, while inadequate liquidities may cause market frictions, which give rise to mispricing. As described in [8], when $l_1$ and $l_2$ are low, a mispricing process $X(t)$ will take longer to go back to the zero mean. This is consistent with the prevailing view that more friction accompanies high illiquidity.

2.2. Consumption habit and wealth process

Assume that the agent exhibits consumption habit formation. Let $h(t)$ be the habit level at time $t$ and satisfy

\[h(t) = h_0 e^{-\xi_1 t} + \xi_1 \int_0^t e^{-\xi_2 (t-s)} c(s) \, ds\]  

or

\[dh(t) = (\xi_1 c(t) - \xi_2 h(t)) \, dt,\]

where $h(0) = h_0 (> 0), \xi_1 (> 0)$ and $\xi_2 (> 0)$ are the initial habit level, the scaling parameter and the persistence parameter respectively. $\xi_1$ measures the intensity of consumption habit, which means that as $\xi_1$ increases, the habit has an emphasis on the past consumption. $\xi_2$ measures the persistence of past consumption. $c(s)$ is the consumption rate at time $t$. Similar to [7, 18, 12, 19], we let $\xi_2 > \xi_1$ ensure that the habit level will decay when the consumption rate is in line with the habit level. In this sense, given $h(t)$, from time $t$ onwards, if the agent’s consumption is exactly at the minimum, that is $c(s) = h(s)$ for $s \geq t$, then the future habit level is given by

\[h(s) = h(t)e^{-\xi_2(s-t)},\]  

for $s \geq t$. 

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Eq.(2.7) shows the minimum consumption level of the agent. Noticed that when \( \xi_2 - \xi_1 \) is small, the consumption habit only descends very slowly even with minimum consumption and thus restricts the agent more (seeing, Kraft et al. [11]), and therefore the larger (smaller) \( \xi_2 - \xi_1 \) means the weaker (stronger) consumption habit strength. In other words, \( \xi_2 - \xi_1 \) depicts consumption habit strength by determining to what extent the current level of habits limits future consumption choices. This implies that the agent is motivated to smooth his/her consumption, suggests that past consumption has a significant impact on current and future economic decisions, and reflects the psychological rationality of the agent.

Again assume that, in the horizon \([0, T]\), from the non-financial market, the agent obtains a stream of income at a rate of \( Y(t) \). The dynamics of \( Y(t) \) satisfies

\[
dY(t) = \mu_Y Y(t) dt, \tag{2.8}
\]

where \( \mu_Y > 0 \) is the growth rate of income.

Let \( \pi_m(t), \pi_1(t), \) and \( \pi_2(t) \) be the portfolio weights of wealth invested in the market index and the pair of mispriced stocks. Then \( 1 - \pi_m(t) - \pi_1(t) - \pi_2(t) \) is the proportion of wealth invested in the risk-free asset. Denote \( \pi = \{ (\pi_m(t), \pi_1(t), \pi_2(t)) : 0 \leq t \leq T \} \), then the wealth process \( W(t) \) can be given by

\[
\begin{aligned}
dW(t) &= (1 - \pi_m(t) - \pi_1(t) - \pi_2(t))rW(t) dt + \pi_m(t)W(t) \frac{dS_m(t)}{S_m(t)} \\
&
+ \pi_1(t)W(t) \frac{dS_1(t)}{S_1(t)} + \pi_2(t)W(t) \frac{dS_2(t)}{S_2(t)} + Y(t) dt - c(t) dt \\
&= W(t)(r + \mu_m \hat{\pi}_m(t) - l_1 \pi_1(t) X(t) + l_2 \pi_2(t) X(t)) dt + Y(t) dt - c(t) dt \\
&+ W(t) \pi_m(t) dB(t) + W(t) (\pi_1(t) + \pi_2(t)) \sigma dZ(t) \\
&+ W(t) \pi_1(t) b dZ_1(t) + W(t) \pi_2(t) b dZ_2(t),
\end{aligned} \tag{2.9}
\]

where

\[
\hat{\pi}_m(t) = \pi_m(t) + \beta (\pi_1(t) + \pi_2(t)). \tag{2.10}
\]

2.3. Optimization problem

**Definition 2.1** (Admissible Strategy). A consumption and investment processes pair \((c, \pi) := \{c(t), \pi(t)\}\) is said to be admissible if it satisfies

1. \([c(t)], [\pi(t)]\) are adapted to \([F(t)], \forall t \in [0, T] \);  
2. \( \mathbb{E} [\int_0^T c(t) dt] < \infty, \mathbb{E} [\int_0^T W(t)^2 \| \pi(t) \|^2 dt] < \infty \), where \( \| \cdot \| \) denotes the Euclidean norm of a vector. 
3. Eq.(2.9) has a pathwise unique strong solution for \( \forall (t, w, x, y, h) \in [0, T] \times \mathcal{R} \times \mathcal{R} \times \mathcal{R}^+ \times \mathcal{R}^+ \). 

Let \( \Pi \) denote the set of all admissible strategies.

Now, the optimization problems of the agent is to seek a strategy \((c, \pi) \in \Pi \) to maximize the expected utility as follows

\[
\max_{(c, \pi) \in \Pi} \mathbb{E} \left[ \int_0^T e^{-\rho s} U_1(c(s) - h(s)) ds + e^{-\rho T} e U_2(W(T)) \right], \tag{2.11}
\]
where $\rho$ is the time preference; $\varepsilon$ describes the individual’s weight relative to the utility of terminal wealth; $U_1(\cdot)$ and $U_2(\cdot)$ are strictly increasing, strictly concave, and continuously differentiable utility functions with the following forms

$$U_i'(0+) = \lim_{x \to 0^+} U_i'(x) = \infty, \quad U_i'(\infty) = \lim_{x \to \infty} U_i'(x) = 0.$$  

(2.12)

Eq. (2.11) shows that the agent obtains utility from the following two aspects: $c(t) - h(t)$, the difference between the consumption $c(t)$ and habit level $h(t)$; and $W(T)$, the terminal wealth. Let $\Pi(t)$ be the set of all admissible strategies over the time interval $[t, T]$, we define the value function at time $t$

$$V(t, w, x, y, h) = \max_{(c, \pi) \in \Pi(t)} \mathbb{E} \left[ \int_t^T e^{-\rho(s-t)} U_1(c(s) - h(s)) ds + e^{-\rho(T-t)} \varepsilon U_2(W(T)) \right],$$  

(2.13)

where $\mathbb{E}[-] = \mathbb{E}[-|W(t) = w, X(t) = x, Y(t) = y, h(t) = h]$.

3. Solution of the optimization problems

To reduce the complexity of the model, suppose that the utility functions have the following expression (cf. Shi et al. [19])

$$U_i(x) = \frac{x^{1-\gamma}}{1-\gamma} := U(x), \quad i = 1, 2,$$

(3.1)

where $\gamma (\gamma > 0, \gamma \neq 1)$ is the relative risk aversion coefficient. Under the power utility function, we solve the optimization (2.13). Moreover, we show two special cases, by imposing special values for the some parameters.

3.1. Optimal consumption and investment strategies

For convenience, we introduce $C^{1,2,2,1,1} := C^{1,2,2,1,1}([0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+) = \{\varphi(t, w, x, y, h): \varphi(t, \cdot, \cdot, \cdot, y, h) \text{ is once continuously differentiable for } t \text{ on } [0, T], y \text{ on } \mathbb{R}^+, \text{ and } h \text{ on } \mathbb{R}^+; \varphi(\cdot, w, x, \cdot, \cdot) \text{ is twice continuously differentiable for } w \text{ on } \mathbb{R} \text{ and } x \text{ on } \mathbb{R}\}$. For any function $\varphi(t, w, x, y, h) \in C^{1,2,2,1,1}$, we define a differential operator:

$$\mathcal{A}^c \varphi(t, w, x, y, h) = -\rho \varphi + \varphi_t + (rw + y - c + \mu_m w_\pi - l_1 w_\pi_1 + l_2 w_\pi_2) \varphi_w$$

$$+ (\xi_1 c - \xi_2 h) \varphi_h + \mu_1 y \varphi_y - (l_1 + l_2) x \varphi_x + \frac{1}{2} (\sigma_m^2 \pi_m^2 + \sigma^2 (\pi_1 + \pi_2)^2)$$

$$+ b^2 (\pi_1^2 + \pi_2^2) w^2 \varphi_{ww} + b^2 \varphi_{xx} + b^2 (\pi_1 - \pi_2) w \varphi_{wx}$$

(3.2)

where $\varphi_t, \varphi_w, \varphi_x, \varphi_y, \varphi_h, \varphi_{ww}, \varphi_{xx}$ and $\varphi_{wx}$ represent the partial derivatives of $\varphi(t, w, x, y, h)$ with respect to (w.r.t) the corresponding variables.

By the principle of stochastic dynamic programming, the corresponding Hamilton-Jacobi-Bellman (HJB) equation of the value function (2.13) can be given

$$\max_{(c, \pi) \in \Pi(t)} \left\{ \mathcal{A}^c \varphi(t, w, x, y, h) + U(c - h) \right\} = 0$$

(3.3)
with the boundary condition \( V(T, w, x, y, h) = \varepsilon U(w) \).

In order to solve (3.3), suppose that \( J(t, h, w, x, y) \) is a solution of (3.3) and satisfies the first-order optimal conditions as following

\[
e^* = h + \left( 1 - \xi_1 \right) J_{w}^{-\frac{1}{2}} J_{w}^{-\frac{1}{2}},
\]

\[
\hat{r}^*_m = -\frac{\mu_m}{\sigma_m^2 w} J_w,
\]

\[
\pi^*_1 = \frac{(\sigma^2(1 + l) + b^2 l) J_w}{b^2(2\sigma^2 + b^2)w J_w} - \frac{J_{wx}}{w J_w},
\]

\[
\pi^*_2 = -\frac{(\sigma^2(1 + l) + b^2 l) x J_w}{b^2(2\sigma^2 + b^2)w J_w} + \frac{J_{wx}}{w J_w}.
\]

Substituting (3.4)–(3.7) into (3.3) has

\[
\frac{\gamma}{1 - \gamma} \left( 1 - \xi_1 \right) \frac{J_h}{J_w} \frac{J_{wy}}{J_w} = \frac{\mu_m}{2\sigma_m^2 J_w} + \frac{(l_1 + l_2) x J_w J_{wx}}{J_{ww}} - b^2 \frac{f_{wx}}{J_{ww}}
\]

\[
- \frac{\sigma^2(l_1 + l_2)^2 + b^2(l_1^2 + l_2^2) x^2 f^2_w}{2b^2(2\sigma^2 + b^2) J_{ww}} + J_t + (rw + y - h)J_w + b^2 J_{xx}
\]

\[-(l_1 + l_2) x J_x + \mu Y y J_y + (\xi_1 - \xi_2) h J_h - \rho J = 0.
\]

We try to conjecture that a solution of Eq. (3.8) satisfies the following form

\[
J(t, w, x, y, h) = \frac{1}{1 - \gamma} f^\gamma(t, x)(w + A(t)y - D(t)h)^{1-\gamma}
\]

with \( A(T) = 0, D(T) = 0 \) and \( f(T, x) = e^{\frac{1}{\gamma}} \), and get by calculation

\[
J_t = (1 - \gamma) J \left( \frac{\gamma f_t}{1 - \gamma f} + \frac{A'(t)y - D'(t)h}{w + A(t)y - D(t)h} \right),
\]

\[
J_w = \frac{(1 - \gamma) J}{w + A(t)y - D(t)h},
\]

\[
J_x = \gamma f_J f_t, \quad J_y = \frac{(1 - \gamma) J A(t)}{w + A(t)y - D(t)h}, \quad J_h = \frac{(1 - \gamma) J D(t)}{w + A(t)y - D(t)h},
\]

\[
J_{ww} = -\frac{\gamma(1 - \gamma) J}{(w + A(t)y - D(t)h)^2}, \quad J_{xx} = \gamma(1 - \gamma) J \left( \frac{1}{1 - \gamma} f_x + \left( \frac{f_x}{f} \right)^2 \right),
\]

\[
J_{wx} = \frac{\gamma(1 - \gamma) J f_x}{w + A(t)y - D(t)h} f_t.
\]

where \( f_t, f_x \) and \( f_{xx} \) are the corresponding partial derivatives of \( f(t, x) \). Substituting the above derivatives and (3.9) into (3.8) obtains

\[
A'(t) + (\mu - r) A(t) + 1 = 0,
\]

\[
D'(t) - (r + \xi_2 - \xi_1) D(t) + 1 = 0,
\]

\[
b^2 f_{xx} - \frac{l_1 + l_2}{\gamma} x f_x + f_t - (\lambda_1 - \lambda_2 x^2) f + \eta(t) = 0.
\]
where \( \lambda = \frac{\sigma^2(l_1 + l_2)^2 + b^2(l_1^2 + l_2^2)}{2 \beta^2 (2\sigma^2 + b^2)} \), \( \lambda_1 = r + \frac{\lambda}{\gamma} \), \( \lambda_2 = \frac{\lambda(1-\gamma)}{\gamma} \), and \( \eta(t) = (1 + D(t) \xi_1)^{-\gamma} \). By the boundary condition \( A(T) = 0 \) and \( D(T) = 0 \), it is easy to solve the ordinary differential equations (ODEs) (3.11) and (3.12) and obtain

\[
A(t) = \int_t^T e^{-(r-\eta(u))} \, du, \quad \text{(3.14)}
\]

\[
D(t) = \int_t^T e^{-(r+\xi_2-\xi_1)} \, du. \quad \text{(3.15)}
\]

However, the ODE (3.13) does not seem easy to solve. Inspired by [10], we try a solution of the form

\[
f(t, x) = e^{\frac{1}{2} e^{-\phi_1(T-t)} - \frac{1}{2} \phi_2^2(T-t)x^2} + \int_t^T \eta(u) e^{-\phi_1(u-t)} - \frac{1}{2} \phi_2^2(u-t)x^2 \, du,
\]

where the functions \( \phi_1(t) \) and \( \phi_2(t) \) are determined. Substituting (3.16) into (3.13), we can get

\[
\phi_{2}'(t) + \frac{2}{\gamma} (l_1 + l_2) \phi_2(t) + 2b^2 \phi_2^2(t) + 2\lambda_2 = 0,
\]

\[
\phi_{1}'(t) - b^2 \phi_2(t) - \lambda_1 = 0,
\]

where \( \phi_1(0) = \phi_2(0) = 0 \). We can solve Eqs. (3.17) and (3.18) and obtain

\[
\phi_2(t) = \frac{2\lambda_2(e^{\epsilon_1t} - e^{\epsilon_2t})}{\kappa_2 e^{\epsilon_1t} - \kappa_1 e^{\epsilon_2t}},
\]

\[
\phi_1(t) = b^2 \int_0^t \phi_2(u) \, du + \lambda_1 t,
\]

where \((l_1 + l_2)^2 > 4b^2 \gamma^2 \lambda_2\), \( \kappa_1 = \frac{1}{\gamma} (l_1 + l_2) - \frac{1}{\gamma} \sqrt{(l_1 + l_2)^2 - 4b^2 \gamma^2 \lambda_2} \) and \( \kappa_2 = \frac{1}{\gamma} (l_1 + l_2) + \frac{1}{\gamma} \sqrt{(l_1 + l_2)^2 - 4b^2 \gamma^2 \lambda_2} \).

Now, we summarize the discussions and give the following theorem.

**Theorem 3.1.** Assume that \((l_1 + l_2)^2 > 4b^2 \gamma^2 \lambda_2\) and for any \( t \in [0, T] \), \( X(t) + A(t)Y(t) - D(t)h(t) > 0 \). Under the boundary condition \( J(T, w, x, y, h) = \epsilon U(w) \), a solution of HJB equation (3.3) is given by (3.9), and the candidate optimal consumption is given by

\[
c^\ast(t) = h + \frac{w + A(t)y - D(t)h}{f(t, x)(1 + \xi_1 D(t))^{\frac{1}{2}}},\quad \text{(3.21)}
\]

The candidate optimal proportions of wealth invested in the market index and the pair of mispriced stocks

\[
\pi_{m}^\ast(t) = \frac{1}{\gamma} \left( \frac{\mu_m}{\gamma s^2 - \frac{\beta (l_2 - l_1)}{2 \sigma^2 + b^2}} \right) \frac{w + A(t)y - D(t)h}{w},\quad \text{(3.22)}
\]

\[
\pi_{l}^\ast(t) = \left( \frac{f(t, x)}{f(t, x)} \right) \frac{(\gamma \sigma^2 (l_1 + l_2) + b^2 l_1)}{(\gamma b^2 (2\sigma^2 + b^2))} \frac{w + A(t)y - D(t)h}{w},\quad \text{(3.23)}
\]

For simplicity, this paper mainly considers the case that the differential equation (3.17) has two different real roots.
\[ \pi_2^*(t) = \left( \frac{(\sigma^2(l_1 + l_2) + b^2l_2)x}{\gamma b^2(2\sigma^2 + b^2)} - \frac{f_s(t, x)}{f(t, x)} \right)w + A(t)y - D(t)h. \] (3.24)

Where \( A(t), D(t) \) and \( f(t, x) \) are given by Eqs. (3.14), (3.15) and (3.16), respectively. Besides, the above \( x, y \) and \( h \) are short for \( X(t), Y(t) \) and \( H(t) \) respectively.

**Proof.** Substituting (3.9) into (3.4)–(3.7), together with (2.10), we can obtain the strategies (3.21)–(3.24). \( \Box \)

The next theorem confirms that a solution to the HJB equation (3.3) is verily the solution of (2.13).

**Theorem 3.2.** Assume that \((l_1 + l_2)^2 > 4b^2\gamma^2l_2\) and for any \( t \in [0, T] \), \( W(t) + A(t)Y(t) - D(t)h(t) \geq 0 \). If \( J(t, w, x, y, h) \) is a solution of HJB equation (3.8) with the boundary condition \( J(T, w, x, y, h) = \varepsilon U(x) \), then the value function \( V(t, w, x, y, h) = J(t, w, x, y, h) \), and the optimal strategies are given by Eqs. (3.21)–(3.24).

The proof of Theorem 3.2 is straightforward (see [19]), and we here omit it.

**Remark 1.** We define \( W(t) + A(t)Y(t) - D(t)h(t) \) as the free wealth of the individual, where \( W(t) \) and \( A(t)Y(t) \) represent financial capital and human capital, and \( D(t)h(t) \) being regarded as the consumption habit buffer is the cost of ensuring that future consumption is not lower than the current habit (cf. [18, 11, 19]). The condition \( W(t) + A(t)Y(t) - D(t)h(t) \geq 0 \) ensures that the sum of financial capital and human capital can sustain the minimum consumption level at time \( t \). Consumption habit strength \( (\xi_2 - \xi_1) \) is important and affects the optimal strategies through \( D(t) \) and thus \( f(t, x) \), which the next section will show by numerical examples.

**Remark 2.** Let \( \hat{V}(t, w, x) \) denote the value function without habit formation. One can prove that when \( \xi = 0, \xi_2 = 0, \) and \( h_0 = 0, \) \( \hat{V}(t, w, x) = V(t, w, x, h) \) \( \xi_1 = \xi_2 = h_0 = 0, i.e., \)

\[ \hat{V}(t, w, x) = \frac{1}{1 - \gamma} f(t, x)(w + A(t)y)^{1 - \gamma}, \] (3.25)

where \( \hat{f}(t, x) = f(t, x) \) \( \xi_1 = \xi_2 = h_0 = 0, i.e., \) when \( \xi_1 = \xi_2 = h_0 = 0, \) \( \hat{f}(t, x) = f(t, x). \)

3.2. Special cases

In this subsection, we focus on finding the optimal value function and strategies for the agents who overlook mispricing and those who take the delta neutral arbitrage strategy for granted.

3.2.1. No mispricing

There are no mispricing opportunities in the market, i.e., \( X(t) = 0 \), we suppose that the individual does not have an insight clearly into specific stock opportunities and only invests in one risk-free asset and a market index. Under this case, let \( \bar{c}(t) \) and \( \bar{p}_m(t) \) denote the consumption rate and the portfolio weight invested in the market index, at time \( t \), respectively. Denote by \( \bar{IC}(t) \) the set of all admissible strategies over the time interval \([t, T]\), under no mispricing opportunities, and then we define the value function in this case

\[ \hat{V}(t, w, y, h) = \max_{(c, \bar{p}_m) \in \bar{IC}(t)} \mathbb{E} \left[ \int_t^T e^{-\kappa(s-t)}U_1(c(s) - h(s))ds + e^{-\kappa(T-t)}\varepsilon U_2(W(T)) \right]. \] (3.26)
Similar to Theorem 3.1 and Theorem 3.2, we can solve (3.26). In fact, as the generality of our main results, we may also get the solution of (3.26) by letting \( x = 0 \) and \( l_1 = l_2 = 0 \) in Theorem 3.1. Thus, to avoid cumbersome, we here only give the following results.

**Proposition 1.** Assume that for any \( t \in [0, T] \), \( X(t) + A(t)Y(t) - D(t)h(t) > 0 \). For the optimization problem (3.26) with no mispricing opportunities, the value function is given by

\[
\bar{V}(t, w, y, h) = \frac{1}{1 - \gamma} \bar{f}^\gamma(t)(w + A(t)y - D(t)h)^{1-\gamma},
\]

(3.27)

where \( \bar{f}(t) = e^{\frac{t}{2}}e^{-\lambda_1(T-t)} + \int_t^T e^{-\lambda_1(u-t)}\eta(u)du \). The optimal consumption is given by

\[
\bar{c}^*(t) = h + \frac{w + A(t)y - D(t)h}{\bar{f}(t)(1 + \xi_1D(t))^\frac{1}{2}}.
\]

(3.28)

The optimal weight of wealth invested in the market index is given by

\[
\bar{\pi}^*_m(t) = \frac{\mu_m}{\gamma\sigma^2_m} \frac{w + A(t)y - D(t)h}{w}.
\]

(3.29)

3.2.2. Delta neutral arbitrage strategy

As described in [15], delta neutral arbitrage strategy allows for \( \pi_1(t) = -\pi_2(t) \). Under this setting, we have \( \bar{\pi}_m(t) = \pi_m(t) \) and the HJB equation (3.3) can be written as follows

\[
\rho V = \max_{(c,\pi_m,\pi_1)\in \Pi(t)} \left\{ \frac{1}{1 - \gamma} (c - h)^{1-\gamma} + V_t + (rw + y - c + \mu_mw\pi_m - (l_1 + l_2)xw\pi_1)V_w \\
+ (\xi_1 c - \xi_2 h)V_h + \mu_y V_y - (l_1 + l_2)xV_x + \frac{1}{2} (\sigma^2_m\pi^2_m + 2b^2\pi^2_1)w^2V_{ww}
+b^2V_{xx} + 2b^2\pi_1wV_{wx} \right\}
\]

(3.30)

Similar to Theorem 3.1 and Theorem 3.2, let \( \bar{\lambda}_2 = \frac{(1-\gamma)(l_1+l_2)^2}{4\gamma^2b^2} \), and then we have the following results under delta neutral arbitrage strategy.

**Proposition 2.** Assume that \( (l_1 + l_2)^2 > 4b^2\gamma^2\bar{\lambda}_2 \) and for any \( t \in [0, T] \), \( X(t) + A(t)Y(t) - D(t)h(t) > 0 \). Under delta neutral arbitrage strategy, the value function is given by

\[
\tilde{V}(t, w, x, y, h) = \frac{1}{1 - \gamma} \tilde{f}^\gamma(t,x)(w + A(t)y - D(t)h)^{1-\gamma}.
\]

(3.31)

The optimal consumption is

\[
\tilde{c}^*(t) = h + \frac{w + A(t)y - D(t)h}{\tilde{f}(t,x)(1 + \xi_1D(t))^\frac{1}{2}}.
\]

(3.32)

The optimal proportions of wealth invested in the market index and the pair of mispriced stocks

\[
\tilde{\pi}^*_m(t) = \frac{\mu_m}{\gamma\sigma^2_m} \frac{w + A(t)y - D(t)h}{w},
\]

(3.33)
\[\ddot{\pi}_1(t) = \left(\ddot{f}(t, x) - \frac{(l_1 + l_2)x}{2yb^2}\right)w + A(t)y - D(t)h - \frac{\tilde{f}(t, x)}{w},\] (3.34)

\[\ddot{\pi}_2(t) = -\ddot{\pi}_1(t).\] (3.35)

Where

\[\ddot{f}(t, x) = e^{\frac{1}{2}}e^{-\tilde{\phi}_1(T-t)-\frac{1}{2}\tilde{\phi}_2(T-t)x^2} + \int_t^T \eta(u)e^{-\tilde{\phi}_1(u-t)-\frac{1}{2}\tilde{\phi}_2(u-t)x^2} du,\] (3.36)

\[\tilde{\phi}_1(\tau) = b^2 \int_0^\tau \tilde{\phi}_2(u)du + \lambda_1\tau,\] (3.37)

\[\tilde{\phi}_2(\tau) = \frac{2\lambda_2(e^{\tilde{\kappa}_1\tau} - e^{\tilde{\kappa}_2\tau})}{\tilde{\kappa}_2e^{\tilde{\kappa}_1\tau} - \tilde{\kappa}_1e^{\tilde{\kappa}_2\tau}},\] (3.38)

and \(\tilde{\kappa}_1 = -\frac{1}{\gamma}(l_1 + l_2) - \frac{1}{\gamma} \sqrt{(l_1 + l_2)^2 - 4b^2\gamma^2\lambda_2}\) and \(\tilde{\kappa}_2 = -\frac{1}{\gamma}(l_1 + l_2) + \frac{1}{\gamma} \sqrt{(l_1 + l_2)^2 - 4b^2\gamma^2\lambda_2}\)

**Remark 3.** There is a sufficient condition to be delta neutral in Eqs. (3.23) and (3.24), i.e., “if \(l_1 = l_2\)”.

### 4. Numerical illustrations

This section presents the quantitative effects of mispricing and habit formation. Referring to [15, 8, 19], the investment time horizon \(t \in [0, 20]\) and the other basic parameters are listed in Table 1.

<table>
<thead>
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<th>Table 1. Values of parameters</th>
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<tr>
<td>(r)</td>
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<td>(\iota_1)</td>
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### 4.1. The effects on consumption and investment strategies

In the beginning, let’s give some abbreviation terms:

- **MPHF:** the model involving mispricing and habit formation;
- **DNHF:** the model involving delta neutral strategy and habit formation;
- **MPNHF:** the model involving mispricing and no habit formation;
- **DNNHF:** the model involving delta neutral strategy and no habit formation;
- **NMPHF:** the model without mispricing but involving habit formation;
- **NMPNHF:** the model without both mispricing and habit formation.

Figure 1† graphs consumption as a function of time \(t\). For a smaller \(t\) (corresponds to a longer time horizon), under considering mispricing or delta neutral strategy (see Figure 1(a)), habit formation damps consumption. However, Figure 1(b) shows that habit formation raises consumption when

†In the figures of this paper, the consumption, and the portfolio weights of wealth invested in the money account, bond stock, and index correspond to their expectations, respectively.
ignoring mispricing or taking the delta neutral strategy diminishes consumption. For a larger $t$ (corresponds to a shorter time horizon), consumption increases with time $t$; the effect of habit formation and mispricing on consumption gradually disappears.

![Figure 1](image1.png)

**Figure 1.** Consumption as a function of time $t$.

Figures 2 and 3 show the portfolio weights of wealth invested in the market index, which decreases with time. Figure 2 says that habit formation decreases the portfolio weights of wealth invested in the market index, while Figure 3 discloses that ignoring mispricing or adopting delta neutral strategy will raise it. Meanwhile, Figure 3 also shows that there is almost no difference between ignoring mispricing and adopting delta neutral strategy when investing in the market index.

![Figure 2](image2.png)

**Figure 2.** Proportion of wealth invested in the market index as a function of time $t$.

Figure 4 illustrates the portfolio weights of wealth invested in the pair of mispriced stocks. (i) We find that, in the long run, the effect of habit formation on the stock investment is not obvious; however, for a larger pricing error $x$ (see Figure 6 (b)), habit formation has a significant impact on the pair of stocks. (ii) Due to, in our example, $X(t) = 0.2, l_1 = 0.2, l_2 = 0.4$, which implies that the price of asset 1 is overestimated and asset 2 is underestimated. Therefore, in Figure 4, the agent shorts selling asset 1 and buys asset 2. Meanwhile, $l_2 > l_1$ means that asset 2 has a stronger mispricing correction ability.
Thus we observe that the proportion of wealth bought assets 2 is higher than one sold asset 1. (iii) As $t$ increases, i.e., the time horizon shortens, the portfolio weights of wealth invested in the pair of mispriced stocks decrease. (iv) For the delta neutral strategy, conclusions (i) and (iii) are also valid.

Figure 5 displays consumption as a function of the pricing error $x$. We can get the following conclusion. Habit formation limits consumption. Under considering mispricing, consumption increases over the pricing error, while consumption has little change with pricing error under adopting the delta neutral strategy.

We illustrate the proportions of wealth invested in the market index and pair of stocks as functions of the pricing error in Figure 6. Figure 6 (a) shows that the proportion of wealth invested in the market index decreases with the pricing error and is limited by habit formation. From Figure 6 (b), we can get the following results. On the one hand, as mentioned above, for a larger pricing error $x$, habit formation affects the proportions of wealth invested in the pair of stocks. On the other hand, as the pricing error increases, the proportion of short-selling asset 1 and buying asset 2 also increases. The reason is that increasing pricing errors will expand the statistical arbitrage opportunities between these two mispricing risky assets. This is also the reason why the proportion of wealth invested in the market
4.2. The wealth-equivalent utility loss

Following [3], we respectively define the wealth-equivalent utility loss (denoted by $L^{HF}$, $L^{MP}$, and $L^{DN}$), under three cases, including ignoring habit formation, ignoring mispricing, and adopting the delta-neutral arbitrage strategy. We have

$$L^{HF} = \left(1 - \left(\frac{\tilde{f}(t,x)}{f(t,x)}\right)^{\frac{1}{\gamma_1}}\right) w + A(t) y - D(t) \frac{h}{w}, \quad (4.1)$$

$$L^{MP} = \left(1 - \left(\frac{\tilde{f}(t,x)}{f(t,x)}\right)^{\frac{1}{1-\gamma_2}}\right) w + A(t) y - D(t) h, \quad (4.2)$$

$$L^{DN} = \left(1 - \left(\frac{\tilde{f}(t,x)}{f(t,x)}\right)^{\frac{1}{1-\gamma_3}}\right) w + A(t) y - D(t) h. \quad (4.3)$$
Figure 7. The wealth-equivalent utility loss as functions of time and price error respectively.

Figure 7 illustrates these wealth-equivalent utility losses. From Figures 7 (a) and (d), we observe that $L_{HF} < 0$ for all time $t$ or pricing error $x$, which implies that when one ignores habit formation, the utility loss does not decrease but increases. As mentioned above, under considering mispricing, habit formation damps consumption, leading to the agent’s utility (from consumption) decreasing. Thus if one ignores habit formation, the utility will increase. Figures 7 (b), (c), (e), and (f) show that ignoring mispricing or adopting the delta-neutral arbitrage strategy will lead to utility loss. The utility loss decreases with time $t$ but increases as the pricing error increases.

5. Conclusions and future works

In this work, we formulate an investment and consumption model. The agent can invest the wealth in one risk-free asset, a market index and a pair of stocks with mispricing, in the financial market, and meanwhile consumes his/her income. The agent with habit formation can optimize strategies by investing mispricing assets. The optimization problem is to find the optimal consumption and investment strategies to maximize the expected utility from consumption and terminal wealth. Specifically, the utility from consumption originated from the difference between the consumption and habit level, i.e., from the part of the consumption that exceeds habit level. Theoretically, a verification theorem for the solution of optimization problem is provided; and we obtain optimal value function and optimal strategies for the agents who overlook mispricing and those who take the delta neutral arbitrage strategy for granted. In the numerical analysis, we discuss the impact of habit formation and mispricing on the optimal strategies. Specifically, the paper considers six cases of the model. That is, the case involving mispricing and habit formation, the case involving delta neutral strategy and habit formation, the case involving mispricing and no habit formation, the case involving delta neutral strategy and no habit formation, the case without mispricing but involving habit formation, and the case without both mispricing and habit formation. The results indicate that the existence of mispricing and the consumption...
habits of agents can both affect their financial behaviors. This reminds us that when studying issues involving consumption and investment, we should acknowledge the existence of mispricing and consider the consumption habits of agents. Finally, we respectively define and analyze the wealth-equivalent utility loss under three cases, including ignoring habit formation, ignoring mispricing, and adopting the delta-neutral arbitrage strategy.

There are some possible extensions of our model. On the one hand, to simplify the model and obtain an analytical solution of the optimization problem, and to highlight the roles of mispricing and consumption habit, we assume that the agent’s income is non random. In future work, we may consider the heterogeneity risk of income and present its effect on the optimal strategies. On the other hand, we may consider an ambiguity-averse agent and investigate his/her specific preference for model ambiguity robustness, or consider preference with the time-varying coefficient of risk aversion similar to Lichtenstern et al. [13].

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Conflict of interest

The authors declare there is no conflicts of interest.

References


