COMPACTNESS FOR COMMUTATORS OF CALDERÓN–ZYGMUND SINGULAR INTEGRAL ON WEIGHTED MORREY SPACES

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ABSTRACT. We prove the boundedness and compactness for the iterated commutators of the θ-type Calderón–Zygmund singular integral and its fractional variant on the weighed Morrey spaces.

1. INTRODUCTION

The aim of this paper is to establish some new results focusing on the boundedness and compactness for the iterated commutators of the θ-type Calderón–Zygmund singular integral and its fractional variant on the weighed Morrey spaces. Let us recall some definitions and background. For $0 \leq \alpha < n$, the θ-type Calderón–Zygmund integral operator T_{K_α} is defined by

$$T_{K_\alpha}(f)(x) = \int_{\mathbb{R}^n} K_\alpha(x,y) f(y) dy \;\text{ for }\; x \notin \operatorname{supp} f \tag{1.1}$$

with kernel K_α satisfying the size condition

$$|K_\alpha(x,y)| \leq \frac{C_{K_\alpha}}{|x-y|^{n-\alpha}} \tag{1.2}$$

and a smoothness condition

$$|K_\alpha(x,y) - K_\alpha(z,y)| + |K_\alpha(y,x) - K_\alpha(y,z)| \leq \theta\Big(\frac{|x-z|}{|x-y|}\Big) \frac{1}{|x-y|^{n-\alpha}}, \tag{1.3}$$

for all $|x-y| > 2|x-z|$, where $\theta : [0,1] \to [0,\infty)$ is a modulus of continuity, that is, θ is a continuous, increasing, subadditive function with $\theta(0) = 0$ and satisfies the Dini condition $\int_0^1 \theta(t) \frac{dt}{t} < \infty$.

When $\alpha = 0$, we denote $T_{K_\alpha} = T_K$. If T_K is bounded on $L^2(\mathbb{R}^n)$, then T_K is just the θ-type Calderón–Zygmund operator. When $\alpha \in (0,1)$, the operator T_{K_α} is the θ-type fractional integral operator. Particularly, when $\theta(t) = t^\delta$ for some $\delta > 0$, the operator T_K is the classical Calderón–Zygmund singular integral operator. It was shown in [14, 15] that T_K is bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$.

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When $\alpha \in (0,1)$, we get from (1.2) that $T_{K_\alpha} f \le C_{K_\alpha} I_\alpha |f|$, where I_α is the classical fractional integral operator defined by

$$I_\alpha(f)(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

As an immediate consequence of the boundedness for I_α, we have that T_{K_α} is bounded from $L^p(w^p)$ to $L^q(w^q)$ for $1 < p < q < \infty$, $1/q = 1/p - \alpha/n$ and $w \in A_{p,q}(\mathbb{R}^n)$ (see the definition of $A_{p,q}(\mathbb{R}^n)$ in Section 2).

On the other hand, the investigation on the boundedness and compactness of the commutators has been the subject of many recent papers in harmonic analysis. In 1976, Coifman, Rochberg and Weiss [5] first introduced the following commutator

$$[b,T](f)(x) = bTf(x) - T(bf)(x)$$

with suitable operator T and function b. More precisely, they established the L^p boundedness for $[b,T]$ with T being Riesz transform for $1 < p < \infty$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$. Later on, Uchiyama [22] improved the above result by showing that $[b,T_\Omega]$ with T_Ω being rough singular integral operator with rough kernel $\Omega \in \mathrm{Lip}_1(\mathrm{S}^{n-1})$ is bounded (resp., compact) on $L^p(\mathbb{R}^n)$ for all $p \in (1,\infty)$ if and only if the symbol $b \in \mathrm{BMO}(\mathbb{R}^n)$ (resp., $b \in \mathrm{CMO}(\mathbb{R}^n)$). Here $\mathrm{CMO}(\mathbb{R}^n)$ is the closure of $C_c^\infty(\mathbb{R}^n)$ in the $\mathrm{BMO}(\mathbb{R}^n)$ topology, which coincides with the space of functions of vanishing mean oscillation. Since then, a considerable amount of attention has been paid to study the boundedness and compactness for the commutators of various operators. For examples, see [1,10,11] for the L^p boundedness of the commutators of rough singular integral, [2,9,13,23,24] for the L^p compactness of the commutators of various integral operators. Other interesting works related to this topic are [20,21,25].

In this paper we focus on the commutators of the θ-type integral operators. More precisely, let T_{K_α} be defined in (1.1). For a locally integrable function b defined on $\mathbb{R}^n$, the commutator $[b,T_{K_\alpha}]$ is given by

$$[b,T_{K_\alpha}](f)(x) := b(x)T_{K_\alpha}(f)(x) - T_{K_\alpha}(bf)(x),$$

for suitable functions f. Let $\mathbb{N} = \{0,1,\ldots\}$ and $m \in \mathbb{N} \setminus \{0\}$. The m-th iterated commutator $(T_{K_\alpha})_b^m$ is defined by

$$(T_{K_\alpha})_b^m(f) := [b,(T_{K_\alpha})_b^{m-1}](f), \quad (T_{K_\alpha})_b^1(f) := [b,T_{K_\alpha}](f).$$

For convenience, we denote $(T_{K_\alpha})_b^m = T_{K_\alpha}$ when $m = 0$. Before stating some known results, let us recall some definitions.

Definition 1.1 ($\mathrm{BMO}(\mathbb{R}^n)$ **space**) ([8]). The $\mathrm{BMO}(\mathbb{R}^n)$ space is given by

$$\mathrm{BMO}(\mathbb{R}^n) := \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n): \ \ \|f\|_{\mathrm{BMO}(\mathbb{R}^n)} := \|M^\sharp f\|_{L^\infty(\mathbb{R}^n)} < \infty\},$$

where $M^\sharp f$ is the sharp maximal function, i.e.

$$M^\sharp f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ that contain the given point x.

Definition 1.2 (**$A_p(\mathbb{R}^n)$ weight**) ([18]). A weight is a nonnegative, locally integrable function on $\mathbb{R}^n$ that takes values in $(0,\infty)$ almost everywhere. For $1<p<\infty$, a weight w is said to be in the Muckenhoupt weight class $A_p(\mathbb{R}^n)$ if there exists a positive constant C such that

$$\sup_{Q \text{ cubes in } \mathbb{R}^n} \Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{1-p'}dx\Big)^{p-1} \leq C. \tag{1.4}$$

The smallest constant C in inequality (1.4) is the corresponding A_p constant of w, which is denoted by $[w]_{A_p}$.

Definition 1.3 (**$A_{p,q}(\mathbb{R}^n)$ weight**) ([19]). Let $0<\alpha<n$, $1<p,q<\infty$ and $1/q=1/p-\alpha/n$. A weight w is said to be in the Muckenhoupt weight class $A_{p,q}(\mathbb{R}^n)$ if there exists a positive constant C such that

$$\sup_{Q \text{ cubes in } \mathbb{R}^n} \Big(\frac{1}{|Q|}\int_Q w^q(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w^{-p'}(x)dx\Big)^{q/p'} \leq C. \tag{1.5}$$

The smallest constant C in inequality (1.5) is the corresponding $A_{p,q}$ constant of w, which is denoted by $[w]_{A_{p,q}}$.

Very recently, Guo, Wu and Yang [9] showed that

Theorem A. ([9]) *Let $0\leq\alpha<n$, $m\in\mathbb{N}\setminus\{0\}$, $1<p<q<\infty$, $1/q=1/p-\alpha/n$ and $w\in A_{p,q}(\mathbb{R}^n)$.*

(i) *If $b\in \mathrm{BMO}(\mathbb{R}^n)$, then*

$$\|(T_{K_\alpha})_b^m(f)\|_{L^q(w^q)} \leq C\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{L^p(w^p)},\ \forall f\in L^p(w^p).$$

(ii) *If $b\in \mathrm{BMO}(\mathbb{R}^n)$, then $(T_{K_\alpha})_b^m$ is a compact operator from $L^p(w^p)$ to $L^q(w^q)$.*

The primary motivation of this note is to establish the corresponding results for $(T_{K_\alpha})_b^m$ on weighted Morrey spaces. Let us recall one definition.

Definition 1.1 (**Weighted Morrey spaces**). Let w,v be two weights on $\mathbb{R}^n$. For $1\leq p<\infty$ and $0\leq\beta<1$, the weighed Morrey space $M^{p,\beta}(w,v)$ is defined as

$$M^{p,\beta}(w,v) := \{f\in L^p_{\mathrm{loc}}(w) : \|f\|_{M^{p,\beta}(w,v)}<\infty\},$$

where

$$\|f\|_{M^{p,\beta}(w,v)} := \sup_{B \text{ balls in } \mathbb{R}^n} \Big(\frac{1}{v(B)^\beta}\int_B |f(x)|^p w(x)dx\Big)^{1/p},$$

where the supremum is taken over all balls in $\mathbb{R}^n$.

This type of Morrey spaces was originally introduced by Komori and Shirai [12] who established that the fractional maximal operator M_α with $0<\alpha<n$ is bounded from $M^{p,\beta}(w^p,w^q)$ to $M^{q,q\beta/p}(w^q)$, provided that $1<p<q<\infty$, $1/q=1/p-\alpha/n$ and $w\in A_{p,q}(\mathbb{R}^n)$. When $w=v$, then $M^{p,\beta}(w,v)$ reduces to the classical weighted Morrey space $M^{p,\beta}(w)$, which was also introduced by Komori and Shirai [12] who established the boundedness for the Hardy-Littlewood maximal operator and the Calderón-Zygmund singular integral operator on $M^{p,\beta}(w)$. When $w\equiv 1$, the space $M^{p,\beta}(w)$ reduces to the classical Morrey space $M^{p,\beta}(\mathbb{R}^n)$, which was first introduced by Morrey [17] to study the local behavior of solutions to second order elliptic partial differential equations. In 1991, Di Fazio and Ragusa [6] presented a characterization of $M^{p,\beta}(\mathbb{R}^n)$ boundedness for $[b,T_\Omega]$. Since then, the characterizations of boundedness and compactness of $[b,T]$ on Morrey spaces $M^{p,\beta}(\mathbb{R}^n)$ have been studied by many authors (see [3, 4, 7]).

In this paper we establish the following results.

Theorem 1.1. *Let $m\in\mathbb{N}$, $0\le\alpha<n$, $1<p<q<\infty$, $1/q=1/p-\alpha/n$, $0\le\beta<p/q$ and $w\in A_{p,q}(\mathbb{R}^n)$.*

(i) *If $b\in \mathrm{BMO}(\mathbb{R}^n)$, then*

$$\|(T_{K_\alpha})_b^m(f)\|_{M^{q,q\beta/p}(w^q)}\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^m\|f\|_{M^{p,\beta}(w^p,w^q)},\ \forall f\in M^{p,\beta}(w^p,w^q).$$

(ii) *If $m\in\mathbb{N}\setminus\{0\}$ and $b\in\mathrm{BMO}(\mathbb{R}^n)$, then $(T_{K_\alpha})_b^m$ is a compact operator from $M^{p,\beta}(w^p,w^q)$ to $M^{q,q\beta/p}(w^q)$.*

Remark 1.1. When $\beta=0$, Theorem 1.1 implies Theorem A. There are some examples satisfying the condition of Theorem 1.1, such as $w=|x|^\gamma$ with $\gamma\in(\alpha-\frac{n}{p},n-\frac{n}{p})$. By Lemma 2.2, it is not difficult to verify $|x|^\gamma\in A_{p,q}(\mathbb{R}^n)$ for $0\le\alpha<n$, $1<p<q<\infty$, $1/q=1/p-\alpha/n$ and $\gamma\in(\alpha-\frac{n}{p},n-\frac{n}{p})$.

As an application of Theorem 1.1, we have the corresponding results for the θ-type Calderón–Zygmund operator and its commutators.

Corollary 1.1. *Let $m\in\mathbb{N}$, $1<p<\infty$, $0\le\beta<1$ and $w\in A_p(\mathbb{R}^n)$.*

(i) *If $b\in\mathrm{BMO}(\mathbb{R}^n)$, then*

$$\|(T_K)_b^m(f)\|_{M^{p,\beta}(w)}\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^m\|f\|_{M^{p,\beta}(w)},\ \forall f\in M^{p,\beta}(w).$$

(ii) *If $m\in\mathbb{N}\setminus\{0\}$ and $b\in\mathrm{BMO}(\mathbb{R}^n)$, then $(T_K)_b^m$ is a compact operator from $M^{p,\beta}(w)$ to $M^{p,\beta}(w)$.*

To prove Theorem 1.1, we will give a boundedness criterion of a class of sublinear operators on weighted Morrey spaces, which has interest in their own right.

Theorem 1.2. *Let $m\in\mathbb{N}$, $0\le\alpha<n$, $1<p<q<\infty$, $1/q=1/p-\alpha/n$, $0\le\beta<p/q$ and $w\in A_{p,q}(\mathbb{R}^n)$. Let T_m be a linear or sublinear operator satisfying*

$$|T_m(f)(x)|\le C_1\int_{\mathbb{R}^n}\prod_{j=1}^m|b_j(x)-b_j(y)|\frac{|f(y)|}{|x-y|^{n-\alpha}}dy, \tag{1.6}$$

where $\vec{b}=(b_1,\dots,b_m)$ with each $b_j\in\mathrm{BMO}(\mathbb{R}^n)$. When $m=0$, we denote $T_0=T$. If T_m satisfies

$$\|T_m(f)\|_{L^q(w^q)}\le C_2\prod_{j=1}^m\|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{L^p(w^p)},\ \ \forall f\in L^p(w^p), \tag{1.7}$$

then for any $f\in M^{p,\beta}(w^p,w^q)$,

$$\|T_m(f)\|_{M^{q,q\beta/p}(w^q)}\le C(C_1,C_2,\beta)\prod_{j=1}^m\|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{M^{p,\beta}(w^p,w^q)}. \tag{1.8}$$

We would like to remark that Theorem 1.2 can apply to the multilinear commutator. More precisely, let $m \in \mathbb{N} \setminus \{0\}$ and T_{K_α} be defined as (1.1). For a vector function $\vec{b} = (b_1, \ldots, b_m)$ with each $b_j \in \mathrm{BMO}(\mathbb{R}^n)$, the multilinear commutator $(T_{K_\alpha})^m_{\vec{b}}$ is defined as

$$\begin{aligned}(T_{K_\alpha})^m_{\vec{b}}(f)(x) &:= [b_m, \cdots [b_2, [b_1, T_{K_\alpha}]] \cdots](x)\\ &= \int_{\mathbb{R}^n} \prod_{j=1}^m (b_j(x) - b_j(y)) \frac{f(y)}{|x-y|^{n-\alpha}} f(y)dy.\end{aligned}$$

Clearly, $(T_{K_\alpha})^m_{\vec{b}} = (T_{K_\alpha})^m_b$ if $\vec{b} = (b_1, \ldots, b_m)$ with $b_j = b$ for $1 \le j \le m$. Recently, Guo, Wu and Yang [9] proved that $(T_{K_\alpha})^m_{\vec{b}}$ is bounded from $L^p(w^p)$ to $L^q(w^q)$ for $0 \le \alpha < n$, $1 < p < q < \infty$, $1/q = 1/p - \alpha/n$ and $w \in A_{p,q}(\mathbb{R}^n)$, provided that each $b_j \in \mathrm{BMO}(\mathbb{R}^n)$ for all $1 \le j \le m$ (see [9, Theorem 5.3]). It is clear that $(T_{K_\alpha})^m_{\vec{b}}$ satisfies the condition (1.6). These facts together with Theorem 1.2 and a slight modification of the proof of the compactness part in Theorem 1.1 implies directly the following result.

Corollary 1.2. *Let $m \in \mathbb{N}$, $0 \le \alpha < n$, $1 < p < q < \infty$, $1/q = 1/p - \alpha/n$ and $w \in A_{p,q}(\mathbb{R}^n)$.*

(i) *If $\vec{b} = (b_1, \ldots, b_m)$ with each $b_j \in \mathrm{BMO}(\mathbb{R}^n)$, then*

$$\|(T_{K_\alpha})^m_{\vec{b}}(f)\|_{M^{q,q\beta/p}(w^q)} \le C \prod_{j=1}^m \|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{M^{p,\beta}(w^p,w^q)}$$

holds for all $f \in M^{p,\beta}(w^p, w^q)$.

(ii) *If $\vec{b} = (b_1, \ldots, b_m)$ with each $b_j \in \mathrm{CMO}(\mathbb{R}^n)$, then $(T_{K_\alpha})^m_{\vec{b}}$ is a compact operator from $M^{p,\beta}(w^p, w^q)$ to $M^{q,q\beta/p}(w^q)$.*

The paper is organized as follows. In Section 2 we present some definitions and lemmas, which are the main ingredients of proving our main results. The proofs of Theorems 1.1 and 1.2 will be given in Section 3. We remark that some ideas of our methods are taken from [9, 13, 16], but our methods and techniques are more delicate and complex than that of [9, 13, 16].

Throughout the paper, for any $p \in (1, \infty]$ we let p' denote the conjugate index of p which satisfies $1/p + 1/p' = 1$ (here we set $\infty' = 1$). The letter C will stand for positive constants not necessarily the same one at each occurrence but is independent of the essential variables. For $x = (x_1, \ldots, x_n)$ we set $|x|_\infty = \max_{1 \le i \le n} |x_i|$.

2. Some definitions and lemmas

In order to prove Theorem 1.1, we need the following properties for $A_p(\mathbb{R}^n)$ and $A_{p,q}(\mathbb{R}^n)$ weighs.

Lemma 2.1. ([16]). *Let $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$. Then*

(i) *There exists a constant $\theta \in (0,1)$ such that $w^{1+\theta} \in A_p(\mathbb{R}^n)$. Both θ and $[w^{1+\theta}]_{A_p}$ depend only on n, p and the A_p constant of w.*

(ii) *There exists a constant $\varepsilon \in (0,1)$ such that $w \in A_{p-\varepsilon}(\mathbb{R}^n)$.*

(iii) *The measure $w(x)dx$ is doubling, i.e. for all $\lambda > 1$ we have*

$$\sup_{Q \text{ cubes in } \mathbb{R}^n} \frac{w(\lambda Q)}{w(Q)} \leq [w]_{A_p} \lambda^{np}.$$

(iv) *There exists a constant $\gamma_w > 1$ such that*

$$\inf_{Q \text{ cubes in } \mathbb{R}^n} \frac{w(2Q)}{w(Q)} \geq \gamma_w.$$

(v) *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$, then*

$$\sup_{Q \text{ cubes in } \mathbb{R}^n} \left(\frac{1}{w(Q)} \int_Q |b(x) - b_Q|^p w(x) dx \right)^{1/p} \simeq_{p,[w]_{A_p}} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

Lemma 2.2. ([19]). *Let $0 < \alpha < n$, $1 < p, q < \infty$, $1/q = 1/p - \alpha/n$ and $w \in A_{p,q}(\mathbb{R}^n)$. Then*

(i) *$w^p \in A_p(\mathbb{R}^n)$, $w^q \in A_q(\mathbb{R}^n)$ and $w^{-p'} \in A_{p'}(\mathbb{R}^n)$.*

(ii)

$$\begin{aligned} w \in A_{p,q}(\mathbb{R}^n) &\Leftrightarrow w^q \in A_{q(n-\alpha)/n}(\mathbb{R}^n) \\ &\Leftrightarrow w^q \in A_{1+q/p'}(\mathbb{R}^n) \Leftrightarrow w^{-p'} \in A_{1+p'/q}(\mathbb{R}^n). \end{aligned}$$

For convenience, we always use the weighted Morrey spaces associated to cubes. Let $1 \leq p < \infty$ and $0 \leq \beta < 1$. For two weights w and v defined on $\mathbb{R}^n$, the weighted Morrey space associated to cubes is defined by

$$\widetilde{M}^{p,\beta}(w,v) := \{f \in L^p_{\mathrm{loc}}(v) : \|f\|_{\widetilde{M}^{p,\beta}(w,v)} < \infty\},$$

where

$$\|f\|_{\widetilde{M}^{p,\beta}(w,v)} := \sup_{Q \text{ cubes in } \mathbb{R}^n} \left(\frac{1}{v(Q)^\beta} \int_Q |f(x)|^p w(x) dx \right)^{1/p},$$

where the supremum is taken over all cubes in $\mathbb{R}^n$.

Remark 2.1. If the weight w is doubling, then we have $\widetilde{M}^{p,\beta}(w,v) = M^{p,\beta}(w,v)$, i.e.

$$\|f\|_{\widetilde{M}^{p,\beta}(w,v)} \simeq \|f\|_{M^{p,\beta}(w,v)}, \tag{2.1}$$

which can be seen by the doubling property for w and the following observation

$$Q(x_0, r) \subset B(x_0, \sqrt{n}/2r) \subset Q(x_0, \sqrt{n}r), \quad \forall x_0 \in \mathbb{R}^n, r > 0.$$

To end this section, we shall present the following characterization that a subset in $M^{p,\beta}(w)$ is a strongly pre-compact set, which plays a key role in the proof of compactness part of Theorem 1.1.

Proposition 2.3. ([16]) *Let $1 < p < \infty$, $0 \leq \beta < 1$ and $w \in A_p(\mathbb{R}^n)$. Then a subset $\mathcal{F}$ of $M^{p,\beta}(w)$ is strongly pre-compact set in $M^{p,\beta}(w)$ if $\mathcal{F}$ satisfies the following conditions:*

(i) *$\mathcal{F}$ is bounded, i.e.*

$$\sup_{f \in \mathcal{F}} \|f\|_{M^{p,\beta}(w)} < \infty;$$

(ii) $\mathcal{F}$ *uniformly vanishes as infinity, i.e.*

$$\lim_{N\to+\infty}\|f\chi_{E_N}\|_{M^{p,\beta}(w)}=0,\ \text{ uniformly for all } f\in\mathcal{F},$$

where $E_N=\{x\in\mathbb{R}^n;|x|>N\}$.

(iii) $\mathcal{F}$ *is uniformly translation continuous, i.e.*

$$\lim_{r\to0}\sup_{h\in B(0,r)}\|f(\cdot+h)-f(\cdot)\|_{M^{p,\beta}(w)}=0,\ \text{ uniformly for all } f\in\mathcal{F}.$$

3. Proofs of main results

In this section we present the proofs of Theorems 1.1 and 1.2. We first prove Theorem 1.2.

Proof of Theorem 1.2. Let $f\in\widetilde{M}^{p,\beta_1}(w^p,w^q)$, $\beta\in(0,p/q)$ and $w\in A_{p,q}(\mathbb{R}^n)$. Fix a cube $Q=Q(x_0,r)$. We divide the proof into two parts:

Step 1. Proof of (1.8) **for** $m=0$. By Remark 2.1, to prove (1.8), it is enough to show that

$$\Big(\frac{1}{w^q(Q)^{q\beta/p}}\int_Q|T(f)(x)|^qw^q(x)dx\Big)^{1/q}\le C\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)},\tag{3.1}$$

where $C>0$ is independent of x_0, r.

We write f as $f=f\chi_{2Q}+f\chi_{(2Q)^c}$. Then we have

$$\begin{aligned}&\Big(\frac{1}{w^q(Q)^{q\beta/p}}\int_Q|\,T(f)(x)|^qw^q(x)dx\Big)^{1/q}\\&\le\Big(\frac{1}{w^q(Q)^{q\beta/p}}\int_Q|\,T(f\chi_{2Q})(x)|^qw^q(x)dx\Big)^{1/q}\\&\quad+\Big(\frac{1}{w^q(Q)^{q\beta/p}}\int_Q|\,T(f\chi_{(2Q)^c})(x)|^qw^q(x)dx\Big)^{1/q}\\&=:I_1+I_2.\end{aligned}\tag{3.2}$$

By Lemma 2.2 (i), we have that $w^q\in A_q(\mathbb{R}^n)$. By Lemma 2.1 (iii), we see that $\frac{w^q(2Q)}{w^q(Q)}\le[w^q]_{A_q}2^{nq}$. This together with the condition (1.7) with $m=0$ implies that

$$\begin{aligned}I_1&=\Big(\frac{1}{w^q(Q)^{q\beta/p}}\int_{2Q}|T(f)(x)|^qw^q(x)dx\Big)^{1/q}\\&\le C_2\frac{1}{w^q(Q)^{\beta/p}}\Big(\int_{2Q}|f(x)|^pw^p(x)dx\Big)^{1/p}\\&=C_2\Big(\frac{1}{w^q(Q)^{\beta}}\int_{2Q}|f(x)|^pw^p(x)dx\Big)^{1/p}\\&\le C_2\Big(\Big(\frac{w^q(2Q)}{w^q(Q)}\Big)^{\beta}\frac{1}{w^q(2Q)^{\beta}}\int_{2Q}|f(x)|^pw^p(x)dx\Big)^{1/p}\\&\le C(C_2,n,p,q,\beta,[w^q]_{A_q})\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}.\end{aligned}\tag{3.3}$$

We now estimate I_2. Fix $x \in Q$, we get by the condition (1.6) with $m = 0$ that

$$T(f\chi_{(2Q)^c})(x) \le C_1 \int_{(2Q)^c} \frac{|f(z)|}{|x-z|^{n-\alpha}} dz. \tag{3.4}$$

Note that $|x-z| \ge |x-z|_\infty \ge |z-x_0|_\infty - |x-x_0|_\infty \ge \frac{1}{2}|z-x_0|_\infty$ for $z \in (2Q)^c$. By (3.4), we have

$$\begin{aligned} T(f\chi_{(2Q)^c})(x) &\le 2^{n-\alpha} C_1 \sum_{l=0}^{\infty} \int_{2^l r \le |z-x_0|_\infty < 2^{l+1} r} \frac{|f(z)|}{|z-x_0|_\infty^{n-\alpha}} dz \\ &\le 2^{n-\alpha} C_1 \sum_{l=0}^{\infty} (2^l r)^{\alpha-n} \int_{2^{l+1}Q} |f(z)| dz. \end{aligned} \tag{3.5}$$

Fix $l \in \mathbb{N}$. Using the Hölder's inequality, one has

$$\begin{aligned} \int_{2^{l+1}Q} |f(z)| dz &\le \Big(\int_{2^{l+1}Q} |f(z)|^p w^p(z) dz \Big)^{1/p} \Big(\int_{2^{l+1}Q} w^{-p'}(z) dz \Big)^{1/p'} \\ &\le w^q(2^{l+1}Q)^{\beta/p} \|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)} \Big(\int_{2^{l+1}Q} w^{-p'}(z) dz \Big)^{1/p'}. \end{aligned} \tag{3.6}$$

Since $w \in A_{p,q}(\mathbb{R}^n)$, then

$$\Big(\int_{2^{l+1}Q} w^{-p'}(z) dz \Big)^{1/p'} \le [w]_{A_{p,q}}^{1/q} |2^{l+1}Q|^{1-\frac{\alpha}{n}} w^q(2^{l+1}Q)^{-1/q}. \tag{3.7}$$

Combining (3.7) with (3.6) leads to

$$\int_{2^{l+1}Q} |f(z)| dz \le [w]_{A_{p,q}}^{1/q} |2^{l+1}Q|^{1-\frac{\alpha}{n}} w^q(2^{l+1}Q)^{\frac{q\beta/p-1}{q}} \|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}. \tag{3.8}$$

In light of (3.5) and (3.8) we would have

$$\begin{aligned} T(f\chi_{(2Q)^c})(x) &\le 2^{n-\alpha} C_1 \sum_{l=0}^{\infty} (2^l r)^{\alpha-n} \int_{2^{l+1}Q} |f(z)| dz \\ &\le C(C_1, n, \alpha, [w]_{A_{p,q}}) \|f\|_{\widetilde{M}^{p,\beta_1}(w^p,w^q)} \\ &\quad \times \sum_{l=0}^{\infty} (2^l r)^{\alpha-n} |2^{l+1}Q|^{1-\frac{\alpha}{n}} w^q(2^{l+1}Q)^{\frac{q\beta/p-1}{q}} \\ &\le C(C_1, n, \alpha, [w]_{A_{p,q}}) \|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)} \sum_{l=0}^{\infty} w^q(2^{l+1}Q)^{\frac{q\beta/p-1}{q}}. \end{aligned} \tag{3.9}$$

Note that $q\beta/p < 1$. Invoking Lemma 2.1 (iv) and (3.9), we have

$$\begin{aligned} I_2 &\le C(C_1, n, \alpha, p, q, [w]_{A_{p,q}}) \|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)} \sum_{l=0}^{\infty} \Big(\frac{w^q(2^{l+1}Q)}{w^q(Q)} \Big)^{\frac{q\beta/p-1}{q}} \\ &\le C(C_1, n, \alpha, p, q, [w]_{A_{p,q}}) \|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)} \sum_{l=0}^{\infty} \gamma_{w^q}^{-\frac{(1-q\beta/p)(l+1)}{q}} \\ &\le C(C_1, n, \alpha, p, q, [w]_{A_{p,q}}) \|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}. \end{aligned}$$

This combining with (3.2) and (3.3) implies

$$\begin{aligned}&\Big(\frac{1}{w^q(Q)^{q\beta/p}}\int_Q |T(f)(x)|^q w^q(x)dx\Big)^{1/q}\\ &\leq C(C_1,C_2,n,\alpha,p,q,[w]_{A_{p,q}})\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}.\end{aligned}$$

This proves (3.1) and completes the proof of the case $m=0$.

Step 2: Proof of (1.8) for $m\in\mathbb{N}\setminus\{0\}$. Let $f\in\widetilde{M}^{p,\beta}(w^p,w^q)$ and $\beta\in(0,p/q)$. Fix a cube $Q=Q(x_0,r)$. By Remark 2.1, to prove (1.8) for $m\in\mathbb{N}\setminus\{0\}$, it suffices to show that

(3.10)
$$\Big(\frac{1}{w^q(Q)^{q\beta/p}}\int_Q |T_m(f)(x)|^q w^q(x)dx\Big)^{1/q}\leq C\prod_{j=1}^m\|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)},$$

where $C>0$ is independent of x_0,r and $\vec{b}$.

Decompose f as $f=f\chi_{2Q}+f\chi_{(2Q)^c}$. We can write

$$\begin{aligned}&\Big(\frac{1}{w^q(Q)^{q\beta/p}}\int_Q |T_m f(x)|^q w^q(x)dx\Big)^{1/q}\\ &\leq\Big(\frac{1}{w^q(Q)^{q\beta/p}}\int_Q |T_m(f\chi_{2Q})(x)|^q w^q(x)dx\Big)^{1/q}\\ &\quad+\Big(\frac{1}{w^q(Q)^{q\beta/p}}\int_Q |T_m(f\chi_{(2Q)^c})(x)|^q w^q(x)dx\Big)^{1/q}\\ &=: J_1+J_2.\end{aligned}\tag{3.11}$$

For J_1. By Theorem A, (1.7) and the fact that $\frac{w^q(2Q)}{w^q(Q)}\leq[w^q]_{A_q}2^{nq}$, we have

$$\begin{aligned}J_1&\leq C_2\prod_{j=1}^m\|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)}\frac{1}{w^q(Q)^{\beta/p}}\Big(\int_{2Q}|f(x)|^p w^p(x)dx\Big)^{1/p}\\ &=C_2\prod_{j=1}^m\|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)}\Big(\frac{1}{w^q(Q)^{\beta}}\int_{2Q}|f(x)|^p w^p(x)dx\Big)^{1/p}\\ &\leq C_2\prod_{j=1}^m\|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)}\\ &\quad\times\Big(\Big(\frac{w^q(2Q)}{w^q(Q)}\Big)^{\beta}\frac{1}{w^q(2Q)^{\beta}}\int_{2Q}|f(x)|^p w^p(x)dx\Big)^{1/p}\\ &\leq C(C_2,n,p,q,\beta,[w^q]_{A_q})\prod_{j=1}^m\|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}.\end{aligned}\tag{3.12}$$

For J_2. Fix $x \in Q$. By (1.6) and the fact that $|x-z| \geq \frac{1}{2}|z-x_0|_\infty$ for $z \in (2Q)^c$, one has

$$\begin{aligned}
&|T_m(f\chi_{(2Q)^c})| \\
&\leq C_1 \int_{(2Q)^c} \prod_{j=1}^m |b_j(x)-b_j(z)| \frac{|f(z)|}{|x-z|^{n-\alpha}} dz \\
&\leq C_1 2^{n-\alpha} \sum_{l=0}^\infty \int_{2^l r \leq |z-x_0|_\infty \leq 2^{l+1} r} \prod_{j=1}^m |b_j(x)-b_j(z)| \frac{|f(z)|}{|z-x_0|_\infty^{n-\alpha}} dz \\
&\leq C_1 2^{n-\alpha} \sum_{l=0}^\infty (2^l r)^{\alpha-n} \int_{2^{l+1}Q} |f(z)| \prod_{j=1}^m |b_j(x)-b_j(z)| dz.
\end{aligned}$$

For convenience, we set $E = \{1, ..., m\}$. For any $j \in \{1, 2, \ldots, m\}$ and $l \in \mathbb{N}$, we let $b_{j,2^{l+1}Q} = \frac{1}{|2^{l+1}Q|} \int_{2^{l+1}Q} b_j(z) dz$. Note that

$$\begin{aligned}
\prod_{j=1}^m |b_j(x)-b_j(z)| &\leq \prod_{j=1}^m (|b_j(x)-b_{j,2^{l+1}Q}| + |b_j(z)-b_{j,2^{l+1}Q}|) \\
&\leq \sum_{\tau \subset E} \Big(\prod_{\mu \in \tau} |b_\mu(x)-b_{\mu,2^{l+1}Q}| \Big) \Big(\prod_{\nu \in E \setminus \tau} |b_\nu(z)-b_{\nu,2^{l+1}Q}| \Big).
\end{aligned}$$

Then we have

$$\begin{aligned}
&T_m(f\chi_{(2Q)^c})(x) \\
&\leq 2^{n-\alpha} C_1 \sum_{\tau \subset E} \Big(\prod_{\mu \in \tau} |b_\mu(x)-b_{\mu,2^{l+1}Q}| \Big) \\
&\quad \times \sum_{l=0}^\infty (2^l r)^{\alpha-n} \int_{2^{l+1}Q} |f(z)| \Big(\prod_{\nu \in E \setminus \tau} |b_\nu(z)-b_{\nu,2^{l+1}Q}| \Big) dz.
\end{aligned}$$

Fix $\tau \subset E$. Let $t = \frac{(1+\varepsilon)p'}{(1+\varepsilon)p'-\varepsilon}$. Clearly, $t \in (1, p)$. By Hölder's inequality, we have

$$\begin{aligned}
&\sum_{l=0}^\infty (2^l r)^{\alpha-n} \int_{2^{l+1}Q} |f(z)| \Big(\prod_{\nu \in E \setminus \tau} |b_\nu(z)-b_{\nu,2^{l+1}Q}| \Big) dz \\
&\leq \sum_{l=0}^\infty (2^l r)^{\alpha-n} \Big(\int_{2^{l+1}Q} |f(z)|^t dz \Big)^{1/t} \\
&\quad \times \Big(\int_{2^{l+1}Q} \Big(\prod_{\nu \in E \setminus \tau} |b_\nu(z)-b_{\nu,2^{l+1}Q}| \Big)^{t'} dz \Big)^{1/t'}.
\end{aligned} \tag{3.13}$$

On the other hand, we can choose $\{s_i\}_{i \in E \setminus \tau} \subset (1, \infty)$ such that $\sum_{i \in E \setminus \tau} 1/s_i = 1$. By Hölder's inequality and the property of $\mathrm{BMO}(\mathbb{R}^n)$, one has

$$\begin{aligned}
&\Big(\int_{2^{l+1}Q} \Big(\prod_{\nu \in E \setminus \tau} |b_\nu(z)-b_{\nu,2^{l+1}Q}| \Big)^{t'} dz \Big)^{1/t'} \\
&\leq \prod_{\nu \in E \setminus \tau} \Big(\int_{2^{l+1}Q} |b_\nu(z)-b_{\nu,2^{l+1}Q}|^{s_\nu t'} dz \Big)^{1/(s_\nu t')} \\
&\leq \prod_{\nu \in E \setminus \tau} \|b_\nu\|_{\mathrm{BMO}(\mathbb{R}^n)} |2^{l+1}Q|^{1/(s_\nu t')} \\
&\leq |2^{l+1}Q|^{1/t'} \prod_{\nu \in E \setminus \tau} \|b_\nu\|_{\mathrm{BMO}(\mathbb{R}^n)}.
\end{aligned} \tag{3.14}$$

Let $s = p/t$. Then $1/(s't) = 1/t - 1/p = 1/(p'(1+\varepsilon))$. By Hölder's inequality, one has

$$
\begin{aligned}
&\left(\int_{2^{l+1}Q} |f(z)|^t dz\right)^{1/t} \\
&\le \left(\int_{2^{l+1}Q} |f(z)|^p w^p(z) dz\right)^{1/p} \left(\int_{2^{l+1}Q} w^{-s't}(z) dz\right)^{1/(s't)} \\
&\le w(2^{l+1}Q)^{\beta/p} \|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)} \left(\int_{2^{l+1}Q} w^{-p'(1+\varepsilon)}(z) dz\right)^{1/(p'(1+\varepsilon))}.
\end{aligned}
\tag{3.15}
$$

Since $w \in A_{p,q}(\mathbb{R}^n)$, by Lemma 2.2, we have $w^{-p'} \in A_{1+\frac{p'}{q}}(\mathbb{R}^n)$. By Lemma 2.1 (i), there exist a constant $\varepsilon \in (0,1)$ such that

$$
w^{-p'(1+\varepsilon)} \in A_{1+\frac{p'}{q}}(\mathbb{R}^n) \subset A_{1+\frac{p'(1+\varepsilon)}{q}}(\mathbb{R}^n).
$$

Then we have

$$
\begin{aligned}
&\left(\int_{2^{l+1}Q} w^{-p'(1+\varepsilon)}(z) dz\right)^{1/(p'(1+\varepsilon))} \\
&\le [w^{-p'(1+\varepsilon)}]_{A_{1+\frac{p'(1+\varepsilon)}{q}}}^{\frac{1}{p'(1+\varepsilon)}} |2^{l+1}Q|^{\frac{1}{1+\varepsilon}+\frac{\varepsilon}{p(1+\varepsilon)}-\frac{\alpha}{n}} w^q(2^{l+1}Q)^{-\frac{1}{q}} \|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}.
\end{aligned}
\tag{3.16}
$$

Note that $\frac{1}{1+\varepsilon} + \frac{\varepsilon}{p(1+\varepsilon)} = \frac{1}{t}$. It follows from (3.13)–(3.16) that

$$
\begin{aligned}
&\sum_{l=0}^{\infty} (2^l r)^{\alpha-n} \int_{2^{l+1}Q} |f(z)| \Big(\prod_{\nu\in E\setminus\tau} |b_\nu(z) - b_{\nu,2^{l+1}Q}| \Big) dz \\
&\le \sum_{l=0}^{\infty} (2^l r)^{\alpha-n} [w^{-p'(1+\varepsilon)}]_{A_{1+\frac{p'(1+\varepsilon)}{q}}(\mathbb{R}^n)}^{\frac{1}{p'(1+\varepsilon)}} |2^{l+1}Q|^{\frac{1}{t}-\frac{\alpha}{n}} \\
&\quad \times w^q(2^{l+1}Q)^{\beta/p-1/q} \|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)} |2^{l+1}Q|^{\frac{1}{t'}} \prod_{\nu\in E\setminus\tau} \|b_\nu\|_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\le [w^{-p'(1+\varepsilon)}]_{A_{1+\frac{p'(1+\varepsilon)}{q}}}^{\frac{1}{p'(1+\varepsilon)}} \prod_{\nu\in E\setminus\tau} \|b_\nu\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)} \\
&\quad \times \sum_{l=0}^{\infty} w(2^{l+1}Q)^{\beta/p-1/q}.
\end{aligned}
\tag{3.17}
$$

By (3.17), (3.13), Lemma 2.1 (iv) and the fact that $q\beta/p<1$, one has

$$
\begin{aligned}
J_2 &\le C\sum_{\tau\in E}\prod_{\nu\in E\setminus\tau}\|b_\nu\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}\\
&\quad\times\Big(\frac{1}{w^q(Q)^{q\beta/p}}\int_Q\Big(\sum_{l=0}^{\infty}\Big(\prod_{\mu\in\tau}|b_\mu(x)-b_{\mu,2^{l+1}Q}|\Big)\\
&\quad\times w^q(2^{l+1}Q)^{\frac{q\beta/p-1}{q}}\Big)^q w^q(x)dx\Big)^{1/q}\\
&\le C\sum_{\tau\in E}\prod_{\nu\in E\setminus\tau}\|b_\nu\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}\\
&\quad\times\Big(\frac{1}{w^q(Q)}\int_Q\Big(\sum_{l=0}^{\infty}\Big(\prod_{\mu\in\tau}|b_\mu(x)-b_{\mu,2^{l+1}Q}|\Big)\\
&\quad\times\Big(\frac{w^q(2^{l+1}Q)}{w^q(Q)}\Big)^{\frac{q\beta/p-1}{q}}\Big)^q w^q(x)dx\Big)^{1/q}\\
&\le C\sum_{\tau\in E}\prod_{\nu\in E\setminus\tau}\|b_\nu\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}w^q(Q)^{-1/q}\\
&\quad\times\Big(\int_Q\Big(\sum_{l=0}^{\infty}\gamma_{w^q}^{-\frac{(1-q\beta/p)(l+1)}{q}}\Big(\prod_{\mu\in\tau}|b_\mu(x)-b_{\mu,2^{l+1}Q}|\Big)\Big)^q w^q(x)dx\Big)^{1/q}.
\end{aligned}
\tag{3.18}
$$

We can choose $\{t_i\}_{i\in\tau}\subset(1,\infty)$ such that $\sum_{i\in\tau}1/t_i=1$. By Minkowski's inequality and Hölder's inequality, one has

$$
\begin{aligned}
&\Big(\int_Q\Big(\sum_{l=0}^{\infty}\gamma_{w^q}^{-\frac{(1-q\beta/p)(l+1)}{q}}\Big(\prod_{\mu\in\tau}|b_\mu(x)-b_{\mu,2^{l+1}Q}|\Big)\Big)^q w^q(x)dx\Big)^{1/q}\\
&\le\sum_{l=0}^{\infty}\gamma_{w^q}^{-\frac{(1-q\beta/p)(l+1)}{q}}\Big(\int_Q\Big(\prod_{\mu\in\tau}|b_\mu(x)-b_{\mu,2^{l+1}Q}|\Big)^q w^q(x)dx\Big)^{1/q}\\
&\le\sum_{l=0}^{\infty}\gamma_{w^q}^{-\frac{(1-q\beta/p)(l+1)}{q}}\prod_{\mu\in\tau}\Big(\int_Q\Big(|b_\mu(x)-b_{\mu,2^{l+1}Q}|\Big)^{qt_\mu}w^q(x)dx\Big)^{1/(qt_\mu)}.
\end{aligned}
\tag{3.19}
$$

Note that $w^q\in A_q(\mathbb{R}^n)$. By Lemma 2.1 (v), Minkowski's inequality and the fact that $|b_{\mu,Q}-b_{\mu,2^{l+1}Q}|\le C(l+1)\|b_\mu\|_{\mathrm{BMO}(\mathbb{R}^n)}$, we obtain

$$
\begin{aligned}
&\Big(\int_Q\Big(|b_\mu(x)-b_{\mu,2^{l+1}Q}|\Big)^{qt_\mu}w^q(x)dx\Big)^{1/(qt_\mu)}\\
&\le|b_{\mu,Q}-b_{\mu,2^{l+1}Q}|w^q(Q)^{1/(qt_\mu)}\\
&\quad+\Big(\int_Q\Big(b_\mu(x)-b_{\mu,Q}\Big)^{qt_\mu}w^q(x)dx\Big)^{1/(qt_\mu)}\\
&\le C(l+1)\|b_\mu\|_{\mathrm{BMO}(\mathbb{R}^n)}w^q(Q)^{1/(qt_\mu)}.
\end{aligned}
\tag{3.20}
$$

It follows from (3.18)–(3.20) that

$$
\begin{aligned}
J_2&\le\prod_{j=1}^{m}\|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}\sum_{l=0}^{\infty}\frac{l+1}{\gamma_{w^q}^{\frac{(1-q\beta/p)(l+1)}{q}}}\\
&\le C\prod_{j=1}^{m}\|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}
\end{aligned}
\tag{3.21}
$$

since $\gamma_{w^q} > 1$ and $q\beta/p < 1$. Then (3.10) follows from (3.21), (3.11) and (3.12). This completes the proof of Theorem 1.2. $\square$

Proof of Theorem 1.1. The boundedness part of Theorem 1.1 follows easily from Corollary 1.2. We prove the compactness part of Theorem 1.1 by considering five steps:

Step 1. Reduction via approximation argument. For a fixed $b \in \mathrm{CMO}(\mathbb{R}^n)$ and $\varepsilon \in (0,1)$, there exists $b_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ such that $\|b_\varepsilon - b\|_{\mathrm{BMO}(\mathbb{R}^n)} < \varepsilon$. It is clear that

$$b_\varepsilon^m - b^m = (b_\varepsilon - b)(b_\varepsilon^{m-1} + b_\varepsilon^{m-2} b + \cdots + b^{m-1}).$$

For convenience, we set

$$\vec{b}_1 = (b_\varepsilon - b, \overbrace{b_\varepsilon, \cdots, b_\varepsilon}^{m-1}), \vec{b}_2 = (b_\varepsilon - b, \overbrace{b_\varepsilon, \cdots, b_\varepsilon}^{m-2}, b), \cdots, \vec{b}_m = (b_\varepsilon - b, \overbrace{b, \cdots, b}^{m-1}).$$

We can write

$$|(T_{K_\alpha})^m_{b_\varepsilon}(f)(x) - (T_{K_\alpha})^m_b(f)(x)| \le \sum_{j=1}^m (T_{K_\alpha})^m_{\vec{b}_j}(f)(x),$$

which combining with Corollary 1.2 and Minkowski's inequality implies that

$$\begin{aligned}
&\|(T_{K_\alpha})^m_{b_\varepsilon}(f) - (T_{K_\alpha})^m_b(f)\|_{M^{q,q\beta/p}(w^q)} \\
&\le \sum_{j=1}^m \|(T_{K_\alpha})^m_{\vec{b}_j}(f)\|_{M^{q,q\beta/p}(w^q)} \le C\varepsilon \|f\|_{M^{p,\beta}(w^p,w^q)}.
\end{aligned}$$

This together with [26, p. 278, Theorem (iii)] implies that to obtain the compactness for $(T_{K_\alpha})^m_b$ with $b \in \mathrm{CMO}(\mathbb{R}^n)$, it suffices to prove the compactness for $(T_{K_\alpha})^m_b$ with $b \in C_c^\infty(\mathbb{R}^n)$.

In what follow, we let $b \in (C)_c^\infty(\mathbb{R}^n)$. We want to show that $(T_{K_\alpha})^m_b$ is compact from $M^{p,\beta}(w^p, w^q) \to M^{q,q\beta/p}(w^q)$.

Step 2. Reduction via smooth truncated techniques. We shall adopt the truncated techniques followed from [13] to prove the compactness part. Let $\varphi \in C^\infty([0,\infty))$ satisfy that $0 \le \varphi \le 1$, $\varphi(t) \equiv 1$ if $t \in [0,1]$ and $\varphi(t) \equiv 0$ if $t \in [2,\infty)$. For any $\eta > 0$, we define the function $K_{\alpha,\eta}$ by

$$K_{\alpha,\eta}(x,y) = K_\alpha(x,y)(1 - \varphi(2\eta^{-1}|x-y|)).$$

By (1.2), we have

$$\begin{aligned}
&|\ (T_{K_{\alpha,\eta}})^m_b(f) - (T_{K_\alpha})^m_b(f)| \\
&\le \int_{\mathbb{R}^n} |(b(x) - b(z))^m f(z)||(K_{\alpha,\eta}(x,z) - K_\alpha(x,z))| dz \\
&= \int_{\mathbb{R}^n} |(b(x) - b(z))^m f(z)||K(x,z)|\varphi(2\eta^{-1}|x-z|) dz \qquad (3.22) \\
&\le C_{K_\alpha}(\|b\|_{L^\infty(\mathbb{R}^n)} + |b(x)|)^{m-1} \|\nabla b\|_{L^\infty(\mathbb{R}^n)} \int_{|x-z|\le\eta} \frac{|f(z)|}{|x-z|^{n-\alpha-1}} dz \\
&\le C_{K_\alpha}(\|b\|_{L^\infty(\mathbb{R}^n)} + |b(x)|)^{m-1} \|\nabla b\|_{L^\infty(\mathbb{R}^n)} 2^{n-\alpha} \omega_n \eta M_\alpha(f)(x)
\end{aligned}$$

for every $x \in \mathbb{R}^n$, where $\omega_n = |B(0,1)|$. Here M_α with $0 < \alpha < n$ is the usual fractional maximal operator defined by

$$M_\alpha(f)(x) = \sup_{r>0} \frac{1}{|B(0,r)|^{1-\alpha/n}} \int_{|y|\le r} |f(x-y)|dy.$$

Combining (3.22) with the $M^{p,\beta}(w^p,w^q) \to M^{q,q\beta/p}(w^q)$ boundedness for M_α implies

$$\begin{aligned} &\|(T_{K_{\alpha,\eta}})_b^m(f) - (T_{K_\alpha})_b^m(f)\|_{M^{q,q\beta/p}(w^q)} \\ &\le C\eta \|f\|_{M^{p,\beta}(w^p,w^q)}, \ \forall f \in M^{p,\beta}(w^p,w^q). \end{aligned} \tag{3.23}$$

By (3.23) and [26, p. 278, Theorem (iii)], the compactness for $(T_{K_\alpha})_b^m$ reduces to the compactness for $(T_{K_{\alpha,\eta}})_b^m$ when $\eta > 0$ is small enough. We set

$$\mathcal{F} := \{(T_{K_{\alpha,\eta}})_b^m(f) : \|f\|_{M^{p,\beta}(w^p,w^q)} \le 1\}.$$

To prove the compactness of $(T_{K_{\alpha,\eta}})_b^m$, it is enough to show that $\mathcal{F}$ is pre-compactness when $\eta > 0$ is small enough. By Proposition 2.3, it is enough to verify that $\mathcal{F}$ satisfies conditions (i)-(iii) of Proposition 2.3.

Step 3. A verification for condition (i) of Proposition 2.3. Let $\eta \in (0,1)$. By (3.23) and the boundedness part of Theorem 1.1,

$$\begin{aligned} &\| (T_{K_{\alpha,\eta}})_b^m(f)\|_{M^{q,q\beta/p}(w^q)} \\ &\le \|(T_{K_{\alpha,\eta}})_b^m(f) - (T_{K_\alpha})_b^m(f)\|_{M^{q,q\beta/p}(w^q)} + \|(T_{K_\alpha})_b^m(f)\|_{M^{q,q\beta/p}(w^q)} \\ &\le C\|f\|_{M^{p,\beta}(w^p,w^q)} \le C, \end{aligned}$$

when $\|f\|_{M^{p,\beta}(w^p,w^q)} \le 1$. This yields that $\mathcal{F}$ satisfies condition (i) of Proposition 2.3.

Step 4. A verification for condition (ii) of Proposition 2.3. Assume that $b \in C_0^\infty(\mathbb{R}^n)$ and is supported in a cube $Q = Q(0,r)$. Fix $f \in M^{p,\beta}(w^p,w^q)$ with $\|f\|_{M^{p,\beta}(w^p,w^q)} \le 1$ and $E_N := \{x \in \mathbb{R}^n : |x| > N\}$ with $N \ge \max\{nr, 1\}$. Note that $|z| \le n|z|_\infty \le \frac{1}{2}nr \le \frac{1}{2}N \le \frac{1}{2}|x|$ when $x \in E_N$ and $z \in Q$. Then we have $|x-z| \ge |x| - |z| \ge \frac{1}{2}|x|$ when $x \in E_N$ and $z \in Q$. By (1.2), we have

$$|K_{\alpha,\eta}(x,y)| \le |K_\alpha(x,y)| \le \frac{C_{K_\alpha}}{|x-y|^{n-\alpha}}, \quad \text{for } x \ne y. \tag{3.24}$$

Note that $b(x) = 0$ when $x \in E_N$ since $N \ge nr$. By (3.24), we have

$$\begin{aligned} (T_{K_{\alpha,\eta}})_b^m(f)(x) &\le C_{K_\alpha} \int_{\mathbb{R}^n} \frac{|(b(x)-b(z))^m f(z)|}{|x-z|^{n-\alpha}} dz \\ &\le 2^{n-\alpha} C_{K_\alpha} \|b\|_{L^\infty(\mathbb{R}^n)}^m |x|^{\alpha-n} \int_Q |f(z)|dz \end{aligned} \tag{3.25}$$

for every $x \in E_N$. By the arguments similar to those used to derive (3.8), we have

$$\int_Q |f(z)|dz \le [w]_{A_{p,q}}^{1/q} w^q(Q)^{\beta/p - 1/q} |Q|^{1-\alpha/n} \|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}. \tag{3.26}$$

For a fixed cube $\tilde{Q} = \tilde{Q}(x_0,t)$, we get from (3.25) and (3.26) that

$$
\begin{aligned}
&\frac{1}{w^q(\tilde{Q})^{q\beta/p}} \int_{\tilde{Q}} |(T_{K_{\alpha,\eta}})_b^m(f)(x)\chi_{E_N}(x)|^q w^q(x)dx \\
&\le C_1 w^q(Q)^{q\beta/p-1}|Q|^{q\frac{n-\alpha}{n}} \frac{1}{w^q(\tilde{Q})^{q\beta/p}} \int_{\tilde{Q}\cap E_N} |x|^{-(n-\alpha)q} w^q(x)dx \\
&\le C_1 w^q(Q)^{q\beta/p-1}|Q|^{q\frac{n-\alpha}{n}} \\
&\quad \times \frac{1}{w^q(\tilde{Q})^{q\beta/p}} \sum_{j=0}^{\infty} \int_{\tilde{Q}\cap(B(0,2^{j+1}N)\setminus B(0,2^jN))} |x|^{-(n-\alpha)} w^q(x)dx \\
&\le C_1 w^q(Q)^{q\beta/p-1}|Q|^{q\frac{n-\alpha}{n}} \frac{1}{w^q(\tilde{Q})^{q\beta/p}} \\
&\quad \times \sum_{j=0}^{\infty} (2^jN)^{-(n-\alpha)q} w^q(\tilde{Q}\cap(B(0,2^{j+1}N)\setminus B(0,2^jN))) \\
&\le C_1 w^q(Q)^{q\beta/p-1}|Q|^{q\frac{n-\alpha}{n}} \\
&\quad \times \sum_{j=0}^{\infty} (2^jN)^{-(n-\alpha)q} w^q(\tilde{Q}\cap(B(0,2^{j+1}N)\setminus B(0,2^jN)))^{1-q\beta/p},
\end{aligned}
\tag{3.27}
$$

where $C_1 = (2^{n-\alpha}C_{K_\alpha}\|b\|_{L^\infty(\mathbb{R}^n)}^m \|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)})^q [w]_{A_{p,q}}$. Invoking Lemma 2.2, we see that $w^q \in A_{q\frac{n-\alpha}{n}}(\mathbb{R}^n)$. Applying Lemma 2.1(ii), there exists $\varepsilon > 0$ such that $w^q \in A_{q\frac{n-\alpha}{n}-\varepsilon}(\mathbb{R}^n)$. Then by Lemma 2.1(iii) we have

$$
\begin{aligned}
&w^q(\tilde{Q}\cap(B(0,2^{j+1}N)\setminus B(0,2^jN))) \\
&\le w^q(B(0,2^{j+1}N)) \le w^q(Q(0,2^{j+2}N)) \\
&\le [w^q]_{A_{q\frac{n-\alpha}{n}-\varepsilon}} (2^{j+2}N)^{q(n-\alpha)-n\varepsilon} w^q(Q(0,1)).
\end{aligned}
$$

This together with (3.27) yields that

$$
\begin{aligned}
&\frac{1}{w^q(\tilde{Q})^{q\beta/p}} \int_{\tilde{Q}} |(T_{K_{\alpha,\eta}})_b^m(f)(x)\chi_{E_N}(x)|^q w^q(x)dx \\
&\le C_1 [w^q]_{A_{q\frac{n-\alpha}{n}-\varepsilon}(\mathbb{R}^n)}^{1-q\beta/p} w^q(Q)^{q\beta/p-1}|Q|^{q\frac{n-\alpha}{n}} w^q(Q(0,1))^{1-q\beta/p} \\
&\quad \times \sum_{j=0}^{\infty} (2^jN)^{-(n-\alpha)q} (2^{j+2}N)^{(q(n-\alpha)-n\varepsilon)(1-q\beta/p)} \\
&\le C_1 [w^q]_{A_{q\frac{n-\alpha}{n}-\varepsilon}(\mathbb{R}^n)}^{1-q\beta/p} w^q(Q)^{q\beta/p-1}|Q|^{q\frac{n-\alpha}{n}} w(Q(0,1))^{1-q\beta/p} \\
&\quad \times \sum_{j=0}^{\infty} (2^jN)^{-q^2\beta(n-\alpha)/p-n\varepsilon(1-q\beta/p)} \\
&\le C_1 [w^q]_{A_{q\frac{n-\alpha}{n}-\varepsilon}(\mathbb{R}^n)}^{1-q\beta/p} w^q(Q)^{q\beta/p-1}|Q|^{q\frac{n-\alpha}{n}} w(Q(0,1))^{1-q\beta/p} \\
&\quad \times N^{-q^2\beta(n-\alpha)/p-n\varepsilon(1-q\beta/p)},
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&\|(T_{K_{\alpha,\eta}})_b^m(f)\chi_{E_N}\|_{M^{q,q\beta/p}(w^q)} \\
&\le 2^{n-\alpha}C_{K_\alpha}\|b\|_{L^\infty(\mathbb{R}^n)}^m \|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)} [w]_{A_{p,q}}^{1/q} [w^q]_{A_{q\frac{n-\alpha}{n}-\varepsilon}}^{\frac{1-q\beta/p}{q}} w^q(Q)^{\frac{q\beta/p-1}{q}} |Q|^{\frac{n-\alpha}{n}} \\
&\quad \times w^q(Q(0,1))^{\frac{1-q\beta/p}{q}} N^{-q\beta(n-\alpha)/p-n\varepsilon(1/q-\beta/p)}.
\end{aligned}
$$

This together with (3.27) implies that $\mathcal{F}$ satisfies the condition (ii) of Proposition 2.3.

Step 5. A verification for condition (iii) of Proposition 2.3. It suffices to show that

$$\lim_{|h|\to 0}\|(T_{K_{\alpha,\eta}})_b^m(f)(\cdot+h)-(T_{K_{\alpha,\eta}})_b^m(f)(\cdot)\|_{M^{q,q\beta/p}(w^q)}=0 \tag{3.28}$$

for a fixed $\eta\in(0,1)$.

At first we shall prove that

$$|K_{\alpha,\eta}(x,y)-K_{\alpha,\eta}(z,y)|\le C\tilde{\theta}\Big(\frac{|x-z|}{|x-y|}\Big)\frac{1}{|x-y|^{n-\alpha}} \tag{3.29}$$

for all $|x-y|>2|x-z|$,where $\tilde{\theta}:=\theta(t)+t$ and the constant C is independent of η.

When $|x-y|>2|x-z|$, we consider the following different cases:

Case 1: ($|x-y|\ge\eta$ and $|z-y|\ge\eta$). In this case we have $K_{\alpha,\eta}(x,y)=K_\alpha(x,y)$ and$K_{\alpha,\eta}(z,y)=K_\alpha(z,y)$. This together with (1.3) yields (3.29).

Case 2: ($|x-y|<\eta$ and $|z-y|<\eta$). Without loss of generality we may assume that $|x-y|\ge|z-y|$. It is clear that $|y-z|>\frac{1}{2}|x-y|$. We have

$$\begin{aligned}
&|K_{\alpha,\eta}(x,y)-K_{\alpha,\eta}(z,y)|\\
&\le|K_\alpha(x,y)-K_\alpha(z,y)|+|K_\alpha(x,y)-K_\alpha(z,y)|\varphi(2\eta^{-1}|x-y|)\\
&\quad+|K_\alpha(z,y)||\varphi(2\eta^{-1}|x-y|)-\varphi(2\eta^{-1}|z-y|)|.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&|K_{\alpha,\eta}(y,x)-K_{\alpha,\eta}(y,z)|\\
&\le|K_\alpha(y,x)-K_\alpha(y,z)|+|K_\alpha(y,x)-K_\alpha(y,z)|\varphi(2\eta^{-1}|x-y|)\\
&\quad+|K_\alpha(y,z)||\varphi(2\eta^{-1}|x-y|)-\varphi(2\eta^{-1}|z-y|)|
\end{aligned}$$

Above facts together with (1.2) and (1.3) imply

$$\begin{aligned}
&|K_{\alpha,\eta}(x,y)-K_{\alpha,\eta}(z,y)|+|K_{\alpha,\eta}(y,x)-K_{\alpha,\eta}(y,z)|\\
&\le 2(|K_\alpha(x,y)-K_\alpha(z,y)|+|K_\alpha(y,x)-K_\alpha(y,z)|)\\
&\quad+(|K_\alpha(z,y)|+|K_\alpha(y,z)|)|\varphi(2\eta^{-1}|x-y|)-\varphi(2\eta^{-1}|z-y|)|\\
&\le 2\theta\Big(\frac{|x-z|}{|x-y|}\Big)\frac{1}{|x-y|^{n-\alpha}}+\frac{2C_{K_\alpha}}{|y-z|^{n-\alpha}}|\varphi(2\eta^{-1}|x-y|)-\varphi(2\eta^{-1}|z-y|)|.
\end{aligned}$$

Note that $|\varphi'(t)|\le C\chi_{1\le t\le 2}(t)$ for all $t>0$. Then we have

$$\begin{aligned}
&|\varphi(2\eta^{-1}|x-y|)-\varphi(2\eta^{-1}|z-y|)|\\
&\le\frac{2}{\eta}|\varphi'(t)||x-z|\le C\frac{2}{\eta}\chi_{1\le t\le 2}(t)\le C\frac{4|x-z|}{\eta t}\le C\frac{|x-z|}{|x-y|},
\end{aligned} \tag{3.30}$$

where $t\in(\frac{2}{\eta}|z-y|,\frac{2}{\eta}|x-y|)$. Therefore, we get

$$|K_{\alpha,\eta}(x,y)-K_{\alpha,\eta}(z,y)|+|K_{\alpha,\eta}(y,x)-K_{\alpha,\eta}(y,x)|\le C\widetilde{\theta}\Big(\frac{|x-z|}{|x-y|}\Big)\frac{1}{|x-y|^{n-\alpha}},$$

which gives (3.29) in this case.

Case 3: ($|x-y|\geq\eta$ and $|z-y|<\eta$). In this case we have $K_{\alpha,\eta}(x,y)=K_\alpha(x,y)$ and$|z-y|>\frac{1}{2}|x-y|$ since $|x-y|>2|x-z|$. This together with (1.2), (1.3) and (3.30) implies that

$$\begin{aligned}
&|K_{\alpha,\eta}(x,y)-K_{\alpha,\eta}(z,y)|+|K_{\alpha,\eta}(y,x)-K_{\alpha,\eta}(y,z)|\\
&=|K_\alpha(x,y)-K_{\alpha,\eta}(z,y)|+|K_\alpha(y,x)-K_{\alpha,\eta}(y,z)|\\
&\leq|K_\alpha(x,y)-K_\alpha(z,y)|+|K_\alpha(y,x)-K_\alpha(y,z)|\\
&\quad+(|K_\alpha(z,y)|+|K_\alpha(y,z)|)\varphi(2\eta^{-1}|z-y|)\\
&\leq|K_\alpha(x,y)-K_\alpha(z,y)|+|K_\alpha(y,x)-K_\alpha(y,z)|+(|K_\alpha(z,y)|\\
&\quad+|K_\alpha(y,z)|)|\varphi(2\eta^{-1}|z-y|)-\varphi(2\eta^{-1}|x-y|)|\\
&\leq\theta\Big(\frac{|x-z|}{|x-y|}\Big)\frac{1}{|x-y|^{n-\alpha}}+\frac{2C_{K_\alpha}}{|y-z|^{n-\alpha}}\frac{|x-z|}{|x-y|},
\end{aligned}$$

which proves (3.29) in this case.

Case 4: ($|x-y|<\eta$ and $|z-y|\geq\eta$). The case is similar to Case 3.

In what follows, we set $|h|<\frac{\eta}{8}$ and $\eta\in(0,1)$. By the definition of $(T_{K_{\alpha,\eta}})^m_b$,

$$\begin{aligned}
&|(T_{K_{\alpha,\eta}})^m_b(f)(x+h)-(T_{K_{\alpha,\eta}})^m_b(f)(x)|\\
&\leq\int_{\mathbb{R}^n}|(b(x+h)-b(y))^m(K_{\alpha,\eta}(x+h,y)-K_{\alpha,\eta}(x,y))f(y)|dy\\
&\quad+\int_{\mathbb{R}^n}|((b(x+h)-b(y))^m-(b(x)-b(y))^m)K_{\alpha,\eta}(x,y)f(y)|dy\\
&=:L_1+L_2.
\end{aligned}\tag{3.31}$$

For L_1. Due to $|h|<\frac{\eta}{8}$, then we have $K_{\alpha,\eta}(x+h,y)=K_{\alpha,\eta}(x,y)=0$ when $|x-y|\leq\frac{\eta}{4}$. Moreover, $|x-y|>2|h|$ when $|x-y|>\frac{\eta}{4}$. By (3.29), we have that for almost every $x\in\mathbb{R}^n$,

$$\begin{aligned}
L_1&\leq\int_{|x-y|>\frac{\eta}{4}}|b(x+h)-b(y)|^m|K_{\alpha,\eta}(x+h,y)-K_{\alpha,\eta}(x,y)||f(y)|dy\\
&\leq C\int_{|x-y|>\frac{\eta}{4}}\frac{1}{|x-y|^{n-\alpha}}\widetilde{\theta}\Big(\frac{|h|}{|x-y|}\Big)|f(y)|dy\\
&\leq C\sum_{j=0}^\infty\widetilde{\theta}\Big(\frac{2^{2-j}|h|}{\eta}\Big)\int_{2^{j-2}\eta\leq|x-y|\leq2^{j-1}\eta}\frac{1}{|x-y|^{n-\alpha}}|f(y)|dy\\
&\leq C\sum_{j=0}^\infty\widetilde{\theta}\Big(\frac{2^{2-j}|h|}{\eta}\Big)M_\alpha f(x).
\end{aligned}$$

Note that

$$\begin{aligned}
\sum_{j=0}^\infty\widetilde{\theta}\Big(\frac{2^{2-j}|h|}{\eta}\Big)&\leq\sum_{j=0}^\infty\int_{2^{-j}}^{2^{-j+1}}\frac{\widetilde{\theta}(4t|h|/\eta)}{t}dt\leq C\int_0^2\frac{\widetilde{\theta}(4t|h|/\eta)}{t}dt\\
&\leq C\int_0^{8|h|/\eta}\frac{\widetilde{\theta}(t)}{t}dt<\infty.
\end{aligned}$$

This together with the boundedness for $M_\alpha : M^{p,\beta}(w^p,w^q) \to M^{q,q\beta/p}(w^q)$ implies that

$$
\begin{aligned}
&\|L_1\|_{M^{q,\beta_2}(w^q)}\\
&\leq C\Big(\int_0^{8|h|/\eta} \frac{\widetilde{\theta}(t)}{t}dt\Big)\|M_\alpha(f)\|_{M^{q,q\beta/p}(w^q)} \leq C\Big(\int_0^{8|h|/\eta} \frac{\widetilde{\theta}(t)}{t}dt\Big)\|f\|_{M^{p,\beta}(w^p,w^q)}\\
&\leq C\int_0^{8|h|/\eta} \frac{\widetilde{\theta}(t)}{t}dt,
\end{aligned}
$$

which leads to $\|L_1\|_{M^{q,q\beta/p}(w^q)} \to 0$ as $|h| \to 0$.

Divide the second term L_2 by

$$
\begin{aligned}
L_2 &= \int_{\mathbb{R}^n} |(b(x+h)-b(y))^m - (b(x)-b(y))^m||K_{\alpha,\eta}(x,y)f(y)|dy\\
&= \int_{|x-y|>\eta} |(b(x+h)-b(y))^m - (b(x)-b(y))^m||K_{\alpha,\eta}(x,y)f(y)|dy\\
&\quad + \int_{\eta/2\leq|x-y|\leq\eta} |(b(x+h)-b(y))^m - (b(x)-b(y))^m||K_{\alpha,\eta}(x,y)f(y)|dy\\
&=: L_{2,1}+L_{2,2}.
\end{aligned}
$$

We write

$$
\begin{aligned}
&(b(x+h)-b(y))^m - (b(x)-b(y))^m\\
&= (b(x+h)-b(x)+b(x)-b(y))^m - (b(x)-b(y))^m\\
&= \sum_{i=1}^m C_m^i (b(x+h)-b(x))^i (b(x)-b(y))^{m-i}\\
&= \sum_{i=1}^m C_m^i (b(x+h)-b(x))^i \sum_{j=0}^{m-i} C_{m-i}^j b(x)^j (-b(y))^{m-i-j},
\end{aligned}
$$

where $C_N^r = \frac{N!}{r!(N-r)!}$ for any $r,N \in \mathbb{N}$ with $r \leq N$. Hence, we obtain

$$
\begin{aligned}
L_{2,1} &\leq \sum_{i=1}^\infty C_m^i |b(x+h)-b(x)|^i \sum_{j=0}^{m-i} C_{m-i}^j |b(x)|^j\\
&\quad \times \Big|\int_{|x-y|>\eta} K_\alpha(x,y) b(y)^{m-i-j} f(y)dy\Big|\\
&\leq \sum_{i=1}^m C_m^i |b(x+h)-b(x)|^i \sum_{j=0}^{m-i} C_{m-i}^j |b(x)|^j |T_{K_\alpha}(b^{m-i-j}f)(x)|\\
&\leq C|h||T_{K_\alpha}(f)(x)|.
\end{aligned}
$$

From this and the $M^{p,\beta}(w^p,w^q) \to M^{q,q\beta/p}(w^q)$ boundness of T_{K_α}, we obtain

$$
\|L_{2,1}\|_{M^{q,q\beta/p}(w^q)} \leq C|h|\|T_{K_\alpha}f\|_{M^{q,q\beta/p}(w^q)} \leq C|h|\|f\|_{M^{p,\beta}(w^p,w^q)} \leq C|h|.
$$

On the other hand, one has

$$\begin{aligned}
&\Big|\int_{\eta/2\le|x-y|\le\eta} K_{\alpha,\eta}(x,y)b(y)^{m-i-j}f(y)dy\Big|\\
&\le C\int_{\eta/2\le|x-y|\le\eta}|K_{\alpha,\eta}(x,y)||f(y)|dy\\
&\le C\frac{1}{\eta^{n-\alpha}}\int_{\eta/2\le|x-y|\le\eta}|f(y)|dy\le CM_\alpha(f)(x).
\end{aligned}$$

This leads to

$$\begin{aligned}
L_{2,2}\le&\sum_{i=1}^{i}C_m^i|b(x+h)-b(x)|^i\sum_{j=0}^{m-i}C_{m-i}^j|b(x)|^j\\
&\times\Big|\int_{\eta/2\le|x-y|\le\eta}K_{\alpha,\eta}(x,y)b(y)^{m-i-j}f(y)dy\Big|\\
\le&\, C|h|M_\alpha(f)(x).
\end{aligned}$$

It follows that

$$\|L_{2,2}\|_{M^{q,q\beta/p}(w^q)}\le C|h|\|M_\alpha(f)\|_{M^{q,q\beta/p}(w^q)}\le C|h|\|f\|_{M^{p,\beta}(w^p,w^q)}\le C|h|.$$

It follow from above estimates of $L_1, L_{2,1}, L_{2,2}$ that

$$\|(T_{K_{\alpha,\eta}})_b^m(f)(\cdot+h)-(T_{K_{\alpha,\eta}})_b^m(f)(\cdot)\|_{M^{q,q\beta/p}(w^q)}\to 0$$

as $|h|\to 0$, uniformly for all f with $\|f\|_{M^{p,\beta}(w^p,w^q)}\le 1$. This verifies the condition (iii) of Proposition 2.3. Theorem 1.1 is now proved. □

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