Local existence of solutions to the 2D MHD boundary layer equations without monotonicity in Sobolev space

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Abstract: In this work, we investigate the local existence of the solutions to the 2D magnetohydrodynamic (MHD) boundary layer equations on the half plane by energy methods in weighted Sobolev space. Compared to the existence of solutions to the classical Prandtl equations where the monotonicity condition of the tangential velocity plays an important role, we use the initial tangential magnetic field has a lower bound $\delta > 0$ instead of the monotonicity condition of the tangential velocity.

Keywords: MHD boundary layer equations, the existence of solutions, the energy method, the weighted sobolev space.

Mathematics Subject Classification: 35Q35; 76D10; 76D03; 35M33; 76W05

1. Introduction

The magnetohydrodynamic (MHD) boundary layer system was derived by understanding the high Reynolds number limit to the incompressible viscous MHD system ([3, 5, 23]) in a domain with non-slip boundary when both the Reynolds number and the magnetic Reynolds number have the same order. In this paper, we investigate the local existence of the solutions to the following initial boundary value problem for the 2D MHD system in a periodic domain $\mathbb{R}^2_+ = \{(t, x, y) : t \in [0, T], x \in \mathbb{T}, y \in \mathbb{R}_+\}$, which reads as follows

$$
\begin{aligned}
\partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon - (H^\varepsilon \cdot \nabla) H^\varepsilon + \nabla p^\varepsilon &= \mu \varepsilon \Delta u^\varepsilon, \\
\partial_t H^\varepsilon - \nabla \times (u^\varepsilon \times H^\varepsilon) &= \kappa \varepsilon \Delta H^\varepsilon, \\
\nabla \cdot u^\varepsilon &= 0, & \nabla \cdot H^\varepsilon &= 0,
\end{aligned}
$$

(1.1)

where $\mathbb{T}$ stands for a torus or a periodic domain and $\mathbb{R}_+ = [0, +\infty)$. Here, we suppose the viscosity and resistivity coefficients have the same order of a small parameter $\varepsilon$, $u^\varepsilon = (u^1_\varepsilon, u^2_\varepsilon)$ denotes the velocity vector, $H^\varepsilon = (h^1_\varepsilon, h^2_\varepsilon)$ stands for the magnetic field, and the total pressure $p^\varepsilon = \tilde{p}^\varepsilon + \frac{|H^\varepsilon|^2}{2}$ with $\tilde{p}^\varepsilon$ represents the pressure of the fluid. Parameters $\mu$ and $\kappa$ are the viscosity and heat conductivity coefficients respectively.
The initial data of (1.1) are given by

\[(u_1^0, h_1^0)|_{z=0} = (u_{10}, h_{10}).\]  \hspace{1cm} (1.2)

The no-slip boundary conditions are imposed on the velocity field and the magnetic field

\[(u^f, H^f)|_{y=0} = 0.\]  \hspace{1cm} (1.3)

The far fields boundary conditions

\[\lim_{y \to +\infty} (u_1, b_1) = (U, B).\]  \hspace{1cm} (1.4)

Formally, systems (1.1) yields the incompressible ideal MHD system when \(\varepsilon = 0\). However, there is no match for the tangential velocity between the equations (1.1) and the limiting equations on the boundary value \(y = 0\). This is why a boundary layer forms in the vanishing viscosity and resistivity limit process. To look for the term of system (1.1) whose contribution is essential for the boundary layer, we use the same transform as the one used in [20],

\[t = t, \quad x = x, \quad \tilde{y} = \varepsilon^{-\frac{1}{2}}y,\]

then, set

\[
\begin{cases}
    u_1(t, x, \tilde{y}) = u_1^0(t, x, y), & b_1(t, x, \tilde{y}) = h_1^0(t, x, y), \\
    u_2(t, x, \tilde{y}) = \varepsilon^{-\frac{1}{2}}u_2^0(t, x, y), & b_2(t, x, \tilde{y}) = \varepsilon^{-\frac{1}{2}}h_2^0(t, x, y),
\end{cases}
\]

and

\[p(t, x) = p^f(t, x).\]

Then by taking the limit \(\varepsilon \to 0\), the equations (1.1)-(1.4) are transformed into the following 2D MHD boundary layer equations

\[
\begin{aligned}
&\partial_t u_1 + u_1 \partial_x u_1 + u_2 \partial_y u_1 = b_1 \partial_x b_1 + b_2 \partial_y b_1 + \partial_y^2 u_1 - \partial_x p, \\
&\partial_t b_1 + \partial_x (u_2 b_1 - u_1 b_2) = \partial_y^2 b_1, \\
&\partial_t u_2 + \partial_x u_2 = 0, \qquad \partial_x b_1 + \partial_y b_2 = 0, \\
&(u_1, u_2, b_1, b_2)|_{y=0} = 0, \quad \lim_{y \to +\infty} (u_1, b_1) = (U, B), \\
&(u_1, b_1)|_{z=0} = (u_0, b_0)(x, y).
\end{aligned}
\]  \hspace{1cm} (1.5)

Functions \((U(t, x), B(t, x))\) and \(p(t, x)\) are the values on the boundary of the Euler’s tangential velocity and Euler’s pressure of the outflow, which satisfy the Bernoulli’s law,

\[
\begin{cases}
    \partial_t U + U \partial_x U - B \partial_x B + \partial_x p = 0, \\
    \partial_t B + U \partial_x B - B \partial_x U = 0.
\end{cases}
\]

Before exhibiting the main results in this paper, let us briefly review some known results to the problem (1.5). Especially, when the magnetic field \((b_1, b_2)\) are some constants in (1.5), the system reduces to the classical Prandtl equations which was firstly introduced formally by Prandtl [21] in 1904. This system is the foundation of the boundary layer equations. It describes that the away from the boundary part can be considered as general ideal fluid, but the near a rigid wall part is deeply affected.
by the viscous force. Formally, the asymptotic limit of the Navier-Stokes equations can be denoted by the Prandtl equations within the boundary layer and by the Euler equations away from boundary. About sixty years later, under the monotonicity condition on the tangential velocity field in the normal variable to the boundary, Oleinik [19] proved the local-in-time well-posedness to the 2D Prandtl equations by using the Crocco transformation, which is the first systematic work in strictly mathematics. This result together with an expanded introduction to the boundary layer theory was showed in Oleinik-Samokhin’s classical book [20]. Besides, under the Oleinik’s monotonicity assumption, some authors [2, 17, 31] proved the well-posedness of solution for 2D Prandtl equations by using energy method and constructing a new unknown function to eliminate the difficult term from the convection term. In addition to Oleinik’s monotonicity assumption on the tangential velocity field, Xin and Zhang [30] obtained the existence of global weak solutions to the Prandtl equation when the pressure is favourable ($\partial_x p \leq 0$).

When the velocity field equation is coupled with magnetic field equation, the phenomenon of boundary layer is different since the boundary layers of magnetic field may exist and they are more complicated than the classical Prandtl equations. It should be highlighted that the MHD boundary layer equations are an important problem in investigation of plasma with many known results, see [9, 22, 26]. There are some results in [3, 6] on the so-called Hartmann boundary layer when the magnetic field is transversal to the boundary.

However, we are concerned with the case that the magnetic field is tangent to the boundary in this paper, that is the equations (1.5). There are some results in an analytic framework for the 2D MHD boundary layer equations, Xie and Yang [28] considered the global existence of solutions to the 2D MHD boundary layer equations in the mixed Prandtl and Hartmann regime when initial data is a small perturbation of the Hartmann profile, and got the solution in analytic norm is exponential decay in time. Recently, Liu and Zhang [16] established the global existence and asymptotic decay estimate of solutions to the 2D MHD boundary layer equations with small initial data. Xie and Yang [29] investigated the lifespan of solution to the 2D MHD boundary layer system by using the cancellation mechanism and obtained the lifespan of solution has a lower bound. Liu, Xie and Yang [13] studied the well-posedness of solutions to the 2D MHD in an analytic framework. Moreover, inspired by [8] on the classical Prandtl equations, they proved that if the tangential magnetic field is degenerate sufficiently, then the non-degenerate critical point in the tangential velocity does not prevent the formation of singularity. Chen and Li [4] investigated the well-posedness of the MHD boundary layer equation without resistivity by using the paralinearization methods in Sobolev space. Under the assumption that the tangential component of magnetic fields dominates, Li and Xu [11] proved the existence and uniqueness of solutions to the MHD boundary layer equations without viscosity in Sobolev spaces. So far, in addition the well-posedness of solutions in the Sobolev and analytic frameworks, there are some results on the vanishing limits for the incompressible MHD systems, we refer to [25–27] and the references therein for the recent progress.

Under the assumption that the initial tangential magnetic field has a lower bound $\delta_0 > 0$, there are some results to the 2D MHD boundary layer equations. Liu, Xie and Yang [14] investigated the local existence and uniqueness of solutions in weighed Sobolev space $H^m_*(m \geq 5)$ for the two-dimensional nonlinear MHD boundary layer equations by using energy method. As a continuation of [14], the same authors [15] proved the validity of the Prandtl boundary layer expansion and gave a $L^\infty$ estimate on the error by multi-scale analysis under the assumptions that both the viscosity and resistivity coefficients
with same order and the initial tangential magnetic field on the boundary is not degenerate. Liu, Wang, Xie and Yang [12] proved the local well-posedness to the 2D MHD boundary layer equations in Sobolev spaces and got the linear instability of the 2D MHD boundary layer when the tangential magnetic field is degenerate at one point. Gao, Huang and Yao [7] investigated the local well-posedness of solution in weighted conormal Sobolev spaces to the 2D MHD boundary layer equations with any large initial data by energy methods. Huang, Liu and Yang [10] attained the local well-posedness of solutions to the 2D MHD system in weighted Sobolev spaces by applying the classical iteration scheme.

The main differences between our results and those in [14] are as follows: Liu, Xie and Yang [14] investigated the local existence and uniqueness of solutions in weighed Sobolev space $H^m_k(m \geq 5)$ for the 2D nonlinear MHD boundary layer equations. However, we investigate the local existence of solutions to the 2D MHD equations in weighed Sobolev space $H^4_{k+l}$ by energy method in this paper, which is complenfor the previous results [14]. The monotonicity condition on the velocity field is not needed for the well-posedness of the 2D MHD boundary layer equations in this work, we use the tangential magnetic field have a lower positive bound instead of the monotonicity assumption on the tangential velocity in the normal direction to the boundary. We first get the boundedness of the approximate solutions to the regularized MHD boundary layer equations in $H^4_k$ by calculating the lower order derivative boundary values of variable $y$ for the equations (4.1) and combining with Corollary 5.1 in subsection 5.1. Then, we get the estimates of $D^h(u, h)$ with $|\beta| = 4$ by constructing new unknown function in subsection 5.2. We finally obtain the existence of solution to problem (3.1) in $H^4_{k+l}$.

To investigate the existence of solution to problem (1.5), we encounter some difficulties. Similar to the Prandtl equation, the difficulty of solving problem (1.5) in the Sobolev framework is the loss of $x$-derivative in the terms $u_2\partial_y u_1 - b_2\partial_y b_1$ and $u_2\partial_y b_1 - b_2\partial_y u_1$ in the first and second equations of (1.5), respectively. In other words, $u_2 = -\partial_y^{-1}\partial_x u_1$ and $b_2 = -\partial_y^{-1}\partial_x b_1$ by the divergence-free conditions and the boundary conditions. Thus it creates a loss of the $x$-derivative and a $y$-integration to the $y$-variable. Then the standard energy estimates do not work. To overcome this essential difficulty, inspired by recent results in [7, 14], we only need that the following two new observations which can remove the difficult terms in the convection terms. The first one observation is that $\psi := \partial_y^{-1}b_1$ satisfies

$$\partial_\tau \psi + u_2 b_1 - u_1 b_2 = \partial_y^2 \psi.$$  

Another observation is that under the assumption on the non-degeneracy of $h_1$, we use the following unknown functions to lead the cancellation

$$u_m := \partial_x^m u_1 - \frac{\partial_x^m u_1}{b_1} \partial_x^m \psi, \quad b_m := \partial_x^m b_1 - \frac{\partial_x^m b_1}{b_1} \partial_x^m \psi.$$  

With the help of $(u_m, b_m)$, the difficulties in the analysis on $\partial_x^m u_2 \partial_x u_1 - \partial_x^m b_2 \partial_x b_1$ and $\partial_x^m u_2 \partial_x b_1 - \partial_x^m b_2 \partial_x u_1$ mentioned above can be overcame. The detail of equivalent for $(u_m, b_m)$ and $(\partial_x^m u_1, \partial_x^m b_1)$ in the weighted Sobolev framework will be showed in subsection 5.2.

The paper is arranged as follows. In Section 2, we introduce some notations and main results in this paper. In Section 3, we give the compatibility condition of the MHD boundary layer equations. In Section 4, we prove some propositions of the initial boundary value to the nonlinear regularized MHD boundary layer equations. In Section 5, we derive the existence of the approximate solutions to the MHD boundary layer equations and prove Theorem 2.1.
2. Preliminaries and Main Results

As a preparation, we give some notations. Using the tangential derivative operator

\[ \partial^\beta = \partial^\lambda t \partial^\beta_r, \ \beta = (\beta_1, \beta_2) \in \mathbb{N}^2, \ |\beta| = \beta_1 + \beta_2. \]

and then denoting the derivative operator (in both time and space) by

\[ D^\alpha = \partial^\lambda t \partial^\beta_y, \ \text{for} \ (\beta_1, \beta_2, k) \in \mathbb{N}^3, \ |\alpha| = \beta_1 + \beta_2 + k. \]

Next, we introduce the weighted Sobolev spaces \( H^4_{k+l} \) and Sobolev norms as follows

\[ \|f(t)\|_{H^4_{k+l}}^2 := \sum_{\alpha \leq 4} \|\langle y \rangle^{k+l} D^\alpha f(t, \cdot)\|_{L^2}^2, \]

where \( \langle y \rangle = 1 + y \).

We now state our main result as following.

**Theorem 2.1.** Let \( k \geq \frac{1}{2}, \ l \geq 0 \) be real numbers. Assume the initial data \((u_0, b_0) \in H^4_{k+l}\) satisfy the compatibility conditions up to 6 order. Besides, there exists a small enough \( \delta \in (0, 1) \) such that

\[
\begin{align*}
\|\langle y \rangle^{k+l+1} \partial^\beta_t (u_0, b_0)\|_{L^\infty} &\leq \delta^{-1}, \ \text{for} \ i = 1, 2 \\
b_0(x, y) &\geq \delta.
\end{align*}
\]

Then there exists a \( T := T(\delta, k, l, \|\langle u, b \rangle\|_{H^4_{k+l}}^2) \) such that the initial boundary value problem (1.5) has a classical solution \((u, b)\) satisfying \((u, b) \in H^4_{k+l}\).

Before proving the theorem, we introduce some important inequalities. The following inequalities will be used frequently in this paper, whose proofs are given in [14, 31].

**Lemma 2.2.** For the proper functions \( f, g, h \), the following inequalities hold:

(i) For any \( l \in \mathbb{R}, \ \lambda > \frac{1}{2}, \) an integer \( m \geq 3, \) and any \( \alpha = (\beta, k) \in \mathbb{N}^3, \ \bar{\beta} = (\bar{\beta}_1, \bar{\beta}_2) \in \mathbb{N}^2 \) with \( \alpha + \bar{\beta} \leq m, \)

\[
\|\langle D^\alpha g \cdot \partial^\beta \partial_t h(t, \cdot)\|_{L^2} \leq C \|g\|_{H^m_{k+l}} \|h\|_{H^m_{k+l}}. \]  

(ii) For any \( l \in \mathbb{R}, \) an integer \( m \geq 3, \) and any \( \alpha = (\beta, k) \in \mathbb{N}^3, \ \bar{\alpha} = (\bar{\beta}, \bar{k}) \in \mathbb{N}^3 \) with \( \alpha + \bar{\alpha} \leq m, \)

\[
\|\langle D^\alpha f \cdot \partial^\beta \partial_t g(t, \cdot)\|_{L^2} \leq C \|f\|_{H^m_{l_1}} \|g\|_{H^m_{l_2}}, \ \forall \ l_1 + l_2 = l. \]

(iii) If \( \lambda > -\frac{1}{2} \) and \( \lim_{y \to \infty} f(x, y) = 0, \) then

\[
\|\langle y \rangle^{l_1} f\|_{L^2(\mathbb{R}^2)} \leq C \|\langle y \rangle^{l_1+1} \partial_y f\|_{L^2(\mathbb{R}^2)}. \]
3. The compatibility condition of the MHD boundary layer equations

For the simplicity, we consider the case of a uniform outflow \((U, B) = (1, 1)\) in this work, which implies that the pressure \(p\) is a constant. Thus, the MHD boundary layer equations (1.5) are reduced to

\[
\begin{align*}
\partial_t u_1 + u_1 \partial_x u_1 + u_2 \partial_y u_1 &= b_1 \partial_x b_1 + b_2 \partial_y b_1 + \partial_y^2 u_1, \\
\partial_t b_1 + \partial_x (u_2 b_1 - u_1 b_2) &= \partial_y^2 b_1, \\
\partial_x u_1 + \partial_y u_2 &= 0, \quad \partial_x b_1 + \partial_y b_2 = 0, \\
(u_1, u_2, b_1, b_2)|_{y=0} &= 0, \quad \lim_{y \to +\infty} (u_1, b_1) = (1, 1), \\
(u_1, b_1)|_{t=0} &= (u_0, b_0)(x, y).
\end{align*}
\]

Then, we assume the shear flow \(u^s\) be the solution of the following heat equation

\[
\begin{align*}
\partial_t u^s - \partial_y^2 u^s &= 0, \quad (t, y) \in \mathbb{R}_+ \times \mathbb{R}_+, \\
u^s|_{y=0} &= 0, \quad \text{and} \quad \lim_{y \to +\infty} u^s = 1, \\
u^s|_{t=0} &= u^s_0(y).
\end{align*}
\]

At the moment, we also suppose that

\[
\begin{align*}
u_1 &= u^s + u, \quad b_1 = 1 + b, \\
u_2 &= v, \quad b_2 = g.
\end{align*}
\]

Then the MHD boundary layer equations (3.1) become

\[
\begin{align*}
\partial_t u - \partial_y^2 u + (u^s + u) \partial_x u + v \partial_y u - (1 + b) \partial_x b - g \partial_y b + v \partial_y u^s &= 0, \\
\partial_t b - \partial_y^2 b + (u^s + u) \partial_x b + v \partial_y b - (1 + b) \partial_x u - g \partial_y u - g \partial_y u^s &= 0, \\
u_1|_{t=0} &= u_0 - u^s_0, \quad b_1|_{t=0} = b_0 - 1, \\
(u, v, b, g)|_{y=0} &= 0, \quad \lim_{y \to +\infty} (u, b) = (0, 0).
\end{align*}
\]

Integrating equation (3.4)_2 over \([0, y]\) yields that

\[
\partial_t \int_0^y b \psi + v(1 + b) - (u^s + u)g = \partial_y^2 \int_0^y b \psi,
\]

where we have used the boundary conditions \(b|_{y=0} = v|_{y=0} = g|_{y=0} = 0\).

Define

\[
\psi(t, y) = \int_0^y b \psi,
\]

ones yields

\[
\partial_t \psi + v(1 + b) - (u^s + u)g = \partial_y^2 \psi.
\]

Next, we give the following basic estimates of a shear flow \(u^s\) for the heat equation (3.2), see [31].

**Lemma 3.1.** Let \(u^s(t, y)\) be the solution of (3.2), then for any \(T_1 > 0\), it holds that for \(1 \leq p \leq 6\),

\[
|\partial_y^p u^s(t, y)| \leq c_1(y)^{-k-p+1}, \quad \forall (t, y) \in [0, T] \times \mathbb{R}_+,
\]

where \(c_1 > 0\) depend on \(T_1\).
At this moment, let us state the precise version of the compatibility condition for the nonlinear MHD boundary layer equations (3.4) and give the boundary values.

**Proposition 3.2.** Assume that \((u, b)\) is a smooth solution of the system (3.4), then the initial data \((u_0, b_0)\) have to satisfy the following compatibility conditions up to 6 order:

\[
\begin{align*}
\partial_y^6 u_0(x, 0) &= 0, \quad b_0(x, 0) = 0, \\
\partial_y^7 u_0(x, 0) &= 0, \quad \partial_y^7 b_0(x, 0) = 0, \\
\partial_y^4 u_0(x, 0) &= \partial_x \partial_y^3 u_0(0, 0) + \partial_x \partial_y^3 u_0(0, 0) - \partial_x \partial_y^3 u_0(0, 0) - \partial_x \partial_y^3 b_0(0, x, 0), \\
\partial_y^5 b_0(x, 0) &= \partial_y \partial_x \partial_y^4 u_0(0, x, 0) + \partial_y \partial_x \partial_y^4 u_0(0, x, 0) + \partial_y \partial_x \partial_y^4 u_0(0, x, 0) + 3 \partial_y \partial_x u^0(0, 0) \partial_y \partial_x b(0, x, 0). \tag{3.8}
\end{align*}
\]

Moreover,

\[
\begin{align*}
\partial_y^6 u_0(x, 0) &= -2 \partial_x^2 \partial_y b \partial_y^2 u(0, x, 0) + \partial_x \partial_y^3 u(0, x, 0) \\
&\quad - 2 \partial_x^2 \partial_y b(0, x, 0) \partial_y^2 u(0, 0) + \partial_x \partial_y^2 u(0, x, 0) \partial_y^2 u(0, 0) \\
&\quad - \partial_y^3 b \partial_x \partial_y \partial_x b(0, x, 0) + 2 \partial_x b \partial_y^2 \partial_y u(0, x, 0) \\
&\quad + \sum_{1 \leq j \leq 3} C^4_j \partial_y^j (u^0 + u) \partial_x \partial_y^{4-j} u + \partial_y^j \partial_y^{5-j} u - \partial_y^j (1 + b) \partial_x \partial_y^{4-j} b \\
&\quad - \partial_y^j g \partial_y^{5-j} b + \partial_y^j \partial_y^{5-j} u \tag{3.9}
\end{align*}
\]

and

\[
\begin{align*}
\partial_y^6 b_0(x, 0) &= -(3 \partial_y^j u + 4 \partial_x \partial_y u) \partial_x \partial_y u(0, x, 0) - \partial_x b(4 \partial_x \partial_y^3 u + 2 \partial_y^2 \partial_y^2 b)(0, x, 0) \\
&\quad + (4 \partial_x \partial_y^3 b + 2 \partial_y^2 \partial_y u) \partial_x \partial_y^2 u(0, x, 0) + \partial_x \partial_y b(3 \partial_y^3 u + 4 \partial_y \partial_y b)(0, x, 0) \\
&\quad + 3 \partial_x u \partial_x \partial_y b(0, x, 0) + \partial_x u(4 \partial_x \partial_y^3 b + 2 \partial_y^2 \partial_y u)(0, x, 0) \\
&\quad + \sum_{1 \leq j \leq 3} C^4_j \partial_y^j (u^0 + u) \partial_x \partial_y^{4-j} u - \partial_y^j g \partial_y^{5-j} u \\
&\quad - \partial_y^j (1 + b) \partial_x \partial_y^{4-j} u + \partial_y^j \partial_y^{5-j} b - \partial_y^j g \partial_y^{5-j} u \tag{3.10}
\end{align*}
\]

**Proof.** By virtue of the equations (3.4), and the boundary condition (3.4), then we get

\[
\partial_y^2 u(x, 0) = 0, \quad \partial_y^2 b(x, 0) = 0. \tag{3.11}
\]

Applying the operator \(\partial_y\) on (3.4), respectively, we can derive that

\[
\begin{align*}
\begin{cases}
\partial_x \partial_y u - \partial_y^3 u + \partial_y (u^0 + u) \partial_x \partial_y u + v \partial_y u = \partial_x ((1 + b) \partial_x b + g \partial_y u) + \partial_y (v \partial_x u) = 0, \\
\partial_x \partial_y b - \partial_y^3 b + \partial_y (u^0 + u) \partial_x \partial_y b + v \partial_y b = \partial_x ((1 + b) \partial_x u + g \partial_x u) - \partial_y (v \partial_x u) = 0.
\end{cases}
\end{align*}
\]

Hence, using equations (3.2), and equations (3.4), from the above equations, we infer that

\[
\begin{align*}
\begin{cases}
\partial_x \partial_y u(t, x, 0) = \partial_y^3 u(t, x, 0) + \partial_x \partial_y b(t, x, 0), \\
\partial_x \partial_y b(t, x, 0) = \partial_y^3 b(t, x, 0) + \partial_x \partial_y u(t, x, 0).
\end{cases}
\end{align*}
\]

Differentiating equations (3.4) with respect to \(y\) twice, it follows that

\[
\partial_x \partial_y^2 u - \partial_y^4 u + \partial_y^2 ((u^0 + u) \partial_x \partial_y u + v \partial_y u + v \partial_y u') - \partial_x ((1 + b) \partial_x \partial_x b + g \partial_y^2 b) = 0. \tag{3.13}
\]
Invoking the Leibniz formula, we can deduce that
\[
\partial_t^2((u' + u)\partial_t u + v\partial_t u' + v\partial_t u') \\
= \partial_t^2(u' + u)\partial_t u + \partial_t^2 v\partial_t u + \partial_t^2 v\partial_t u' \\
+ (u' + u)\partial_t \partial_t^2 u + v\partial_t^3 u + v\partial_t^3 u' \\
+ 2\partial_t(u' + u)\partial_t \partial_t u + 2\partial_t v\partial_t^2 u + 2\partial_t v\partial_t^2 u'.
\]

Therefore,
\[
\partial_t^4 u(t, x, 0) = \partial_x \partial_t u \partial_t u(t, x, 0) + \partial_x \partial_t \partial_t^3 u(t, 0) - \partial_x \partial_t b \partial_t b(t, x, 0),
\]
where we used the facts \(\partial_t^2 u'(x, 0) = 0, \ 0 \leq 2i \leq 4.

Differentiating (3.14) with respect to \(t\) and using the equality (3.12)_1, it follows that
\[
\partial_t \partial_t^3 u(t, x, 0) = (\partial_t \partial_t^2 u + \partial_t^2 \partial_t b)\partial_t u(t, x, 0) + \partial_x \partial_t \partial_t^3 u(t, x, 0) - \partial_x \partial_t \partial_t^3 u(t, 0) = 0.
\]

Similar to (3.13), we have the following the results about the magnetic velocity \(b\)
\[
\partial_t \partial_t^3 b - \partial_t^2 \partial_t b + \partial_t^2 ((u' + u)\partial_t b + v\partial_t b - (1 + b)\partial_t u - g\partial_t u) = 0.
\]

However, by a direct calculation, we infer that
\[
\partial_t^2((u' + u)\partial_t b + v\partial_t b - g\partial_t u') \\
= \partial_t^2(u' + u)\partial_t b + \partial_t^2 v\partial_t b - \partial_t^2 g\partial_t u' \\
+ (u' + u)\partial_t \partial_t^2 b + v\partial_t^3 b - g\partial_t^3 u' \\
+ 2\partial_t(u' + u)\partial_t \partial_t b + 2\partial_t v\partial_t^2 b - 2\partial_t g\partial_t^2 u'.
\]

and
\[
\partial_t^2((1 + b)\partial_t u + g\partial_t u) = \partial_t^2((1 + b)\partial_t u + \partial_t^2 g\partial_t u + (1 + b)\partial_t^2 \partial_t u + g\partial_t^3 u \\
+ 2\partial_t((1 + b)\partial_t \partial_t u + 2\partial_t g\partial_t^2 u).
\]

Therefore, we can arrive at
\[
\partial_t^4 b(t, x, 0) = -3\partial_t b(t, x, 0)\partial_t \partial_t u(t, x, 0) + 3\partial_x \partial_t b(t, x, 0)\partial_t u(t, 0) \\
+ 3\partial_t(u'(t, 0))\partial_t \partial_t b(t, x, 0).
\]

Differentiating (3.17) with respect to \(t\) and using equality (3.12)_2, it follows that
\[
\partial_t \partial_t^3 b(t, x, 0) = -3(\partial_t^3 b + \partial_x \partial_t u)\partial_t \partial_t u(t, x, 0) - 3\partial_t b(\partial_t \partial_t^3 u + \partial_t^3 \partial_t b)(t, x, 0)
\]
\[ +3(\partial_t \partial_x^2 b + \partial_x \partial_t^2 u)\partial_x \partial_t u(t, x, 0) + 3\partial_t \partial_x b(\partial_t^2 u + \partial_x \partial_t b)(t, x, 0) \\
+3\partial_x \partial_t^2 u \partial_t \partial_x b(t, x, 0) + 3\partial_t \partial_x u(\partial_x \partial_t^2 b + \partial_x^2 \partial_t u)(t, x, 0). \]

(3.18)

Differentiating the equations (3.4)_1 with respect to \( y \) four times, it follows that

\[ \partial_y \partial_y^4 u + \partial_y^5 (u^t + u)\partial_y u + v \partial_y u - (1 + b)\partial_y b - g \partial_y b + v \partial_y u^t \] \( = \partial_y^5 u, \]

using the Leibniz formula again

\[ \partial_y^5 u(t, x, 0) \]
\[ = \partial_y \partial_y^4 u(t, x, 0) - \partial_y \partial_y^3 \partial_x u \partial_x u(t, x, 0) + \partial_y \partial_y^2 \partial_x^2 b \partial_x u(t, x, 0) - \partial_y \partial_y^3 u(t, x, 0) \partial_x u^t(t, 0) \\
- \partial_y \partial_y^4 b(t, x, 0) + \sum_{1 \leq j \leq 3} C_j^l(\partial_y^4 (u^t + u)\partial_y \partial_y^{4-j} u \partial_y v \partial_y^{j-1} u - \partial_y^4 (1 + b)\partial_x \partial_y^{4-j} b \\
- \partial_y^4 g \partial_y^{5-j} b + \partial_y^5 v \partial_y^{j-1} u^t) \] \( = \) \( -2\partial_y^2 \partial_x^2 b \partial_x u(t, x, 0) + \partial_x \partial_y \partial_x \partial_y^3 u(t, x, 0) \]
\[ -2\partial_y^2 \partial_y b(\partial_x, 0) \partial_x u(t, 0) + \partial_y \partial_x \partial_x u(t, x, 0) \partial_y^3 u(t, 0) \\
- \partial_y^3 b \partial_x \partial_x b(t, x, 0) + 2\partial_x \partial_x \partial_y \partial_x u(t, x, 0) \\
+ \sum_{1 \leq j \leq 3} C_j^l(\partial_y^2 (u^t + u)\partial_x \partial_y^{2-j} u + \partial_y^2 v \partial_y^{j-1} u - \partial_y^3 (1 + b)\partial_x \partial_y^{2-j} b \\
- \partial_y^3 g \partial_y^{3-j} b + \partial_y^4 v \partial_y^{j-1} u^t) \] \( = \) \( \partial_y^3 u \).

(3.21)

Analogously, derivating (3.4)_2 with respect to \( y \) four times, it follows

\[ \partial_y \partial_y^3 b + \partial_y^4 (u^t + u)\partial_y b - g \partial_y b - (1 + b)\partial_y u + v \partial_y b - g \partial_y u^t \] \( = \partial_y^4 b, \]

using the Leibniz formula again

\[ \partial_y^4 (u^t + u)\partial_y b - g \partial_y b - (1 + b)\partial_y u + v \partial_y b - g \partial_y u^t \]
\[ = \partial_y^4 (u^t + u)\partial_y b - \partial_y^3 g \partial_y b - \partial_y^4 (1 + b)\partial_x \partial_y^2 u + \partial_y^4 v \partial_y b - g \partial_y^3 b \\
+ (u^t + u)\partial_y \partial_x^2 b - g \partial_x^2 b - (1 + b)\partial_x \partial_x \partial_y b + v \partial_x^2 b - g \partial_x^3 b \\
+ \sum_{1 \leq j \leq 3} C_j^l(\partial_y^3 (u^t + u)\partial_x \partial_y^{3-j} b - \partial_y^3 g \partial_y^{3-j} b - \partial_y^4 (1 + b)\partial_x \partial_y^{3-j} b \\
- \partial_y^4 g \partial_y^{4-j} b + \partial_y^5 v \partial_y^{j-1} u^t) \]
Therefore, it follows from (3.18) and (3.22)-(3.23) that

\[
\partial^6_y b(t, x, 0) = -3(3\partial^3_y b + 4\partial_y \partial_t u \partial_y \partial_t u(t, x, 0) - \partial_y b(4\partial^3_y u + 2\partial^5_y b)](t, x, 0) \\
+ (4\partial_y \partial^5_y b + 2\partial_y \partial_t u \partial_y \partial_t u(t, x, 0)) + \partial_y b(3\partial^3_y u + 4\partial_y \partial_t b)](t, x, 0) \\
+ 3\partial^3_y u \partial_y \partial_t b(t, x, 0) + \partial_y b(4\partial^3_y b + 2\partial^5_y \partial_y \partial_t u)](t, x, 0) \\
+ \sum_{1 \leq j \leq 3} C^j \partial^3_j(u^e + u) \partial_y \partial^5_y - \partial^3_j g \partial^5_y - \partial^3_j (1 + b) \partial_y \partial^5_y u \\
+ \partial^3_j v \partial^5_y - \partial^3_j (1 + b) \partial^5_y - \partial^3_j (1 + b) \partial^5_y u)](x, 0),
\]

(3.24)

then we take the value at \( t = 0 \) for (3.11) (3.14), (3.17), (3.21) and (3.24) can obtain the desired results.

4. Nonlinear regularized MHD boundary layer equations

To investigate the existence of solution of the MHD boundary layer, we consider a parabolic regularized system for problem (3.4), which we can attain the local existence of the solution by using classical energy methods. More specifically, we discuss the following nonlinear MHD systems, for \( 0 < \varepsilon < 1 \),

\[
\begin{align*}
\partial_t u^e - \varepsilon \partial_y^2 u^e - \partial_y^2 u^e + (u^e + u^e) \partial_y u^e + v^e \partial_y u^e - (1 + b^e) \partial_y b^e - g^e \partial_y b^e + v^e \partial_y u^e &= 0, \\
\partial_t b^e - \varepsilon \partial_y^2 b^e - \partial_y^2 b^e + (u^e + u^e) \partial_y b^e + v^e \partial_y b^e - (1 + b^e) \partial_y u^e - g^e \partial_y u^e - g^e \partial_y b^e &= 0, \\
(u^e, b^e)|_{t=0} &= (u^0_0, b^0_0) = (u_0, b_0) + \varepsilon(\mu^1_1, \mu^1_2), \\
(u^e, v^e, b^e, g^e)|_{t=0} &= 0, \lim_{y \to \pm \infty} (u^e, b^e) &= 0,
\end{align*}
\]

(4.1)

where we can use the system (4.1) to construct the corrector terms \( \varepsilon(\mu^1_1, \mu^1_2) \) such that the initial data \( (u_0, b_0) + \varepsilon(\mu^1_1, \mu^1_2) \) satisfy the compatibility conditions up to 6 order for the regularized systems (4.1). We show the boundary data of the solution for the regularized system (4.1) which also give the accurate edition of the compatibility conditions for the system (4.1).

**Proposition 4.1.** Let \( k \geq \frac{1}{2}, l \geq 0 \) be real numbers. Assume that \( (u_0, b_0) \) satisfy the compatibility conditions (3.8)-(3.10) for the equations (3.4), and \( (\mu^1_1, \mu^1_2) \in H^k_{k+1} \) such that \( (u_0, b_0) + \varepsilon(\mu^1_1, \mu^1_2) \) satisfy the compatibility conditions up to 6 order for the regularized system (4.1). If \( (u^e, b^e) \) is a solution to
the problem (4.1) in \([0, T]\) and satisfies \((u^\varepsilon, b^\varepsilon) \in L^\infty([0, T]; H^4_{k,x})\), then we have

\[
\begin{align*}
&\begin{cases}
u^\varepsilon(0, x, 0) = 0, \quad b^\varepsilon(0, x, 0) = 0, \quad x \in \mathbb{R}, \\ \\
\partial^2_x u^\varepsilon(0, x, 0) = 0, \quad \partial^2_y b^\varepsilon(0, x, 0) = 0, \\ \\
\partial^4_x u^\varepsilon(0, x, 0) = \partial_x \partial^3_x u^\varepsilon(0, x, 0) + \partial_x \partial^2_y b^\varepsilon(0, x, 0) - \partial_x \partial_y b^\varepsilon(0, x, 0), \\ \\
\partial^4_y b^\varepsilon(0, x, 0) = -3 \partial_x b^\varepsilon \partial_y u^\varepsilon(0, x, 0) + \partial_x \partial^2_y b^\varepsilon(0, x, 0) \partial_x u^\varepsilon(0, x, 0) + 3 \partial_x u^\varepsilon(0, x, 0) \partial_y b^\varepsilon(0, x, 0) + \epsilon \partial_x \partial^3_y u^\varepsilon(0, x, 0) - 2 \partial_x \partial^2_y b^\varepsilon(0, x, 0) \partial_x \partial_y u^\varepsilon(0, x, 0) - 2 \partial_x \partial^2_y b^\varepsilon(0, x, 0) \partial_x \partial_y b^\varepsilon(0, x, 0) + 2 \epsilon \partial^2_x \partial^2_y b^\varepsilon(0, x, 0) - 2 \epsilon \partial^2_x \partial^2_y b^\varepsilon(0, x, 0) \\
\end{cases}
\end{align*}
\]

(4.2)

Proof. Looking back to the boundary condition in (4.1)

\[
\begin{cases}
u^\varepsilon(t, x, 0) = 0, \quad \nu^\varepsilon(t, x, 0) = 0, \quad b^\varepsilon(t, x, 0) = 0, \\ \\
g^\varepsilon(t, x, 0) = 0, \quad (t, x) \in [0, T] \times \mathbb{R},
\end{cases}
\]

thus the following results are obvious

\[
\partial_x \partial^2_x (u^\varepsilon, b^\varepsilon)(t, x, 0) = 0, \quad \partial_x \partial^2_y (\nu^\varepsilon, g^\varepsilon)(t, x, 0) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}, 0 \leq n \leq 4.
\]

(4.3)

Applying (4.1) and the boundary conditions (4.3), we have

\[
\partial^2_x u^\varepsilon|_{y=0} = 0, \quad \partial^2_y b^\varepsilon|_{y=0} = 0.
\]

(4.4)

Besides, we can also derive

\[
\partial^2_x \partial^2_x u^\varepsilon|_{y=0} = 0, \quad \partial^2_x \partial^2_y b^\varepsilon|_{y=0} = 0,
\]

(4.5)

Deriving the equation of (4.1) with respect to \(y\),

\[
e \partial^2_{xy} \partial_y u^\varepsilon = \partial_x \partial_y u^\varepsilon - \partial^3_y u^\varepsilon + \partial_x ((u^\varepsilon + u^\varepsilon) \partial_x u^\varepsilon + \nu^\varepsilon \partial_y u^\varepsilon + \nu^\varepsilon \partial_y u^\varepsilon) - (1 + b^\varepsilon) \partial_x \partial_y b^\varepsilon - g \partial^2_y b^\varepsilon,
\]

(4.6)

using the boundary conditions (4.4), then we deduce

\[
\partial_x \partial_y|_{y=0} = \partial^3_y \partial_x u^\varepsilon|_{y=0} + \partial_x \partial_y b^\varepsilon|_{y=0} + e \partial^2_x \partial_y u^\varepsilon|_{y=0}.
\]

(4.7)
Similarly, deriving the equation of (4.1) with respect to $y$

$$
\partial_t \partial_y b^f - \partial_y^3 b^f + \partial_y^2 \left( \left( (u^e + u^f) \partial_y b^f + v^f \partial_y b^f - (1 + b^f) \partial_y u^f - g^f \partial_y u^f - g \partial_y u^f \right) \right)
= \epsilon \partial_y^2 \partial_y b^f,
$$

(4.8)

using the boundary condition (4.4) again, then we get

$$
\partial_t \partial_y b^f|_{y=0} = \partial_y^3 b^f|_{y=0} + \partial_y \partial_y u^f|_{y=0} + \epsilon \partial_y^2 \partial_y b^f|_{y=0}.
$$

(4.9)

Differentiating (4.6) with respect to $y$, it follows

$$
e \partial_y^2 \partial_y u^e = \partial_y \partial_y^3 u^e - \partial_y^4 u^e + \partial_y^3 \left( (u^e + u^e) \partial_y u^e + v^e \partial_y u^e + \partial_y u^e \right)
- \partial_y \left( (1 + b^f) \partial_y \partial_y b^f + \epsilon \partial_y^2 b^f \right),
$$

(4.10)

applying Leibniz formula

$$
\partial_y^3 \left( (u^e + u^e) \partial_y u^e + v^e \partial_y u^e + \partial_y u^e \right)
= \partial_y^2 \partial_y u^e + \partial_y^3 \partial_y u^e
+ \epsilon \partial_y^2 \partial_y u^e + \epsilon \partial_y^3 u^e
+ 2 \partial_y (u^e + u^e) \partial_y \partial_y u^e
+ 2 \partial_y v^e \partial_y^2 u^e
+ 2 \partial_y v^e \partial_y^3 u^e.
$$

(4.11)

Therefore, we can derive

$$
\partial_y^3 u^e|_{y=0} = \partial_y \partial_y u^e \partial_y u^e|_{y=0} + \partial_y \partial_y \partial_y u^e \partial_y u^e(t, 0) - \partial_y \partial_y \partial_y b^f \partial_y b^f|_{y=0},
$$

(4.12)

where we used the facts $\partial_y^2 u^e(x, 0) = 0, 0 \leq 2t \leq 4$.

Differentiating (4.12) with respect to $t$ and using (4.7) and (4.9), it follows that

$$
\partial_t \partial_y^3 u^e|_{y=0} = \partial_y \partial_y^3 u^e \partial_y u^e|_{y=0} + \partial_y \partial_y^3 \partial_y u^e \partial_y u^e(t, 0) - \partial_y \partial_y^2 b^f \partial_y^3 \partial_y b^f|_{y=0}.
$$

(4.13)

Analogously, we can arrive at

$$
\partial_y^4 b^f|_{y=0} = -3 \partial_y b^f \partial_y \partial_y u^e|_{y=0} + 3 \partial_y \partial_y b^f|_{y=0} \partial_y u^e(t, 0) + 3 \partial_y u^e(t, 0) \partial_y \partial_y b^f|_{y=0},
$$

(4.14)

and

$$
\partial_t \partial_y^4 b^f|_{y=0} = -3 \partial_y b^f \partial_y \partial_y u^e|_{y=0} + 3 \partial_y b^f|_{y=0} \partial_y u^e(t, 0) - 3 \partial_y b^f \partial_y \partial_y b^f|_{y=0}
+ 3 \partial_y \partial_y b^f \partial_y \partial_y \partial_y u^e|_{y=0} + 3 \partial_y \partial_y b^f \partial_y \partial_y b^f|_{y=0}
+ 3 \partial_y \partial_y b^f \partial_y \partial_y u^e|_{y=0} + 3 \partial_y \partial_y b^f \partial_y \partial_y b^f|_{y=0}
+ 3 \partial_y \partial_y b^f \partial_y \partial_y b^f|_{y=0}.
$$

(4.15)
Differentiating (4.1) with respect to $y$ four times, it follows

$$
\partial_y^4 u^e + \epsilon \partial_y^2 \partial_y^4 u^e = \partial_y \partial_y^3 u^e + \partial_y^4((u^e + u^e)\partial_y u^e + v^e \partial_y u^e)
$$

$$
-(1 + b^e)\partial_y b^e - g^e \partial_y b^e + v^e \partial_y u^e,
$$

(4.16)

using the Leibniz formula again

$$
\partial_y^4((u^e + u^e)\partial_y u^e + v^e \partial_y u^e - (1 + b^e)\partial_y b^e + \epsilon \partial_y b^e + v^e \partial_y u^e)
$$

$$
+(u^e + u^e)\partial_y \partial_y^3 u^e + v^e \partial_y^3 u^e - (1 + b^e)\partial_y \partial_y^3 b^e - g^e \partial_y b^e + v^e \partial_y b^e
$$

$$
+ \sum_{1 \leq j \leq 3} C^4_j(\partial_j^2(u^e + u^e)\partial_y \partial_y^{j-1} u^e + \partial_j^3 v^e \partial_y \partial_y^{j-1} u^e - \partial_j^4(1 + b^e)\partial_y \partial_y^{j-1} b^e
$$

$$
- \partial_j^4 g^e \partial_y \partial_y^{j-1} b^e + \partial_j^4 v^e \partial_y \partial_y^{j-1} u^e)\big|_{y=0}
$$

(4.17)

Hence, using (4.12) and (4.14), we have

$$
\partial_y^4 u^e\big|_{y=0} = -2 \partial_y^2 \partial_y^2 \partial_y^4 u^e\big|_{y=0} + \partial_y \partial_y^3 \partial_y^4 u^e\big|_{y=0}
$$

$$
-2 \partial_y^2 \partial_y^3 \partial_y^4 u^e\big|_{y=0} + \partial_y \partial_y^4 \partial_y^4 u^e\big|_{y=0}
$$

$$
- \partial_y \partial_y^4 \partial_y^4 u^e\big|_{y=0} + 2 \partial_y \partial_y^4 \partial_y^4 u^e\big|_{y=0}
$$

$$
+ 2 \epsilon \partial_y \partial_y^3 \partial_y^4 u^e\big|_{y=0} - 2 \epsilon \partial_y \partial_y \partial_y^4 \partial_y^4 u^e\big|_{y=0}
$$

$$
+ \sum_{1 \leq j \leq 3} C^4_j(\partial_j^4(u^e + u^e)\partial_y \partial_y^{j-1} u^e + \partial_j^4 v^e \partial_y \partial_y^{j-1} u^e - \partial_j^4(1 + b^e)\partial_y \partial_y^{j-1} b^e
$$

$$
- \partial_j^4 g^e \partial_y \partial_y^{j-1} b^e + \partial_j^4 v^e \partial_y \partial_y^{j-1} u^e)\big|_{y=0}.
$$

(4.18)

Similarly, derivating (4.1) with respect to $y$ four times, it follows

$$
\partial_y^4 b^e + \epsilon \partial_y^2 \partial_y^4 b^e = \partial_y \partial_y^3 b^e + \partial_y^4((u^e + u^e)\partial_y b^e - g^e \partial_y b^e
$$

$$
-(1 + b^e)\partial_y b^e + v^e \partial_y b^e - g^e \partial_y u^e),
$$

(4.19)

using the Leibniz formula again

$$
\partial_y^4((u^e + u^e)\partial_y b^e - g^e \partial_y b^e - (1 + b^e)\partial_y u^e + v^e \partial_y b^e - g^e \partial_y u^e)
$$
Corollary 4.1. Assume that \((u_0^\varepsilon, b_0^\varepsilon)\) satisfy the compatibility conditions (4.2) for the equations (4.1) and \((u_0^\varepsilon, b_0^\varepsilon) \in L^\infty([0, T]; H^4)\), then for any \(0 < \varepsilon < 1\), there exists \((\mu_1^\varepsilon, \mu_2^\varepsilon) \in H^6\) such that \((u_0 + \varepsilon \mu_1^\varepsilon, b_0 + \varepsilon \mu_2^\varepsilon)\) satisfy the compatibility conditions up to 6 order for the regularized system (4.1).

Thus, using (4.9), (4.15) and (4.20), we derive

\[
\frac{\partial_j}{\partial y} b_j^{\varepsilon} |_{y=0} = -(3\partial_j^3 b^\varepsilon + 4\partial_j \partial_j \partial_j u^\varepsilon)\partial_j \partial_j \partial_j u^\varepsilon |_{y=0} - \partial_j b (4\partial_j \partial_j^2 u^\varepsilon + 2\partial_j^2 \partial_j b^\varepsilon) |_{y=0} + (4\partial_j \partial_j^3 b^\varepsilon + 2\partial_j^2 \partial_j u^\varepsilon)\partial_j \partial_j \partial_j u^\varepsilon |_{y=0} + \partial_j \partial_j \partial_j u^\varepsilon (4\partial_j \partial_j^2 b^\varepsilon + 2\partial_j^2 \partial_j \partial_j u^\varepsilon) |_{y=0} + 6\varepsilon (\partial_j^3 \partial_j u^\varepsilon \partial_j \partial_j b^\varepsilon - \partial_j^2 \partial_j \partial_j b^\varepsilon \partial_j \partial_j u^\varepsilon) |_{y=0} \]

\[
+ \sum_{1 \leq j \leq 3} C_j \left( \partial_j (u^\varepsilon + u^\varepsilon) \partial_j \partial_j^4 b^\varepsilon - \partial_j \partial_j^3 b^\varepsilon \partial_j^2 \partial_j ^4 u^\varepsilon - \partial_j (1 + b^\varepsilon) \partial_j \partial_j^4 \partial_j u^\varepsilon \right) \quad (4.21)
\]

Similar to (3.8) and (3.9), we can deduce the desired results. Besides, we can see that the equalities (3.21) and (3.24) are different from (4.18) and (4.21), respectively. It is obviously that the underlined terms are new terms. The proof is thus completed. □

According to the relational expressions of the compatibility conditions \((u_0, b_0)\) and \((u_0^\varepsilon, b_0^\varepsilon)\), respectively, we can also obtain the expression of the corrector terms \(\partial_j^3 (\mu_1^\varepsilon, \mu_2^\varepsilon)\), \((0 \leq i \leq 3)\). Thus, we have the following corollary.

**Corollary 4.1.** Assume that \((u_0^\varepsilon, b_0^\varepsilon)\) satisfy the compatibility conditions (4.2) for the equations (4.1) and \((u_0^\varepsilon, b_0^\varepsilon) \in L^\infty([0, T]; H^4)\), then for any \(0 < \varepsilon < 1\), there exists \((\mu_1^\varepsilon, \mu_2^\varepsilon) \in H^6\) such that \((u_0 + \varepsilon \mu_1^\varepsilon, b_0 + \varepsilon \mu_2^\varepsilon)\) satisfy the compatibility conditions up to 6 order for the regularized system (4.1),

\[
\|u_0^\varepsilon\|_{H^6_{u,v}(\mathbb{R}^2)} + \|b_0^\varepsilon\|_{H^6_{u,v}(\mathbb{R}^2)} \leq \frac{3}{2} (\|u_0\|_{H^6_{u,v}(\mathbb{R}^2)} + \|b_0\|_{H^6_{u,v}(\mathbb{R}^2)}),
\]

and

\[
\lim_{\varepsilon \to 0} \|(u_0^\varepsilon, b_0^\varepsilon) - (u_0, b_0)\|_{H^6_{u,v}(\mathbb{R}^2)} = 0.
\]

**Proof.** We use the proof of the Proposition 4.1 to prove this corollary. Taking the value at \(t = 0\) for (4.5). Then we have the following functions \((\mu_1^\varepsilon, \mu_2^\varepsilon)\) from (4.5)

\[
(\partial_j^3 \partial_j \mu_1^\varepsilon, \partial_j^3 \partial_j \mu_2^\varepsilon) = 0, \quad x \in \mathbb{R}
\]

and

\[
\partial_j^4 u_0(x, 0) + \varepsilon \partial_j^4 \mu_1^\varepsilon(x, 0)
\]
\[ \begin{align*}
&= (\partial_s \partial_t u_0 + \varepsilon \partial_s \partial_t \mu_1^s)(\partial_s u_0 + \varepsilon \partial_s \partial_t \mu_1^s)(x, 0) + \partial_s u^s(\partial_s \partial_t u_0 + \varepsilon \partial_s \partial_t \mu_1^s)(x, 0) \\
&\quad - (\partial_s b_0 + \varepsilon \partial_s \partial_t \mu_2^s)(\partial_s \partial_t b_0 + \varepsilon \partial_s \partial_t \mu_2^s)(x, 0). \\
\end{align*} \]

Therefore, we get

\[ \begin{align*}
\partial_s^4 \mu_1^s(x, 0) &= \partial_s \partial_t u_0 \partial_s \partial_t \mu_1^s(x, 0) + \partial_s u_0 \partial_s \partial_t \mu_1^s(x, 0) + \varepsilon \partial_s \partial_t \mu_1^s \partial_s \partial_t \mu_1^s(x, 0) \\
&\quad + \partial_s \partial_t u_0^s(0) \partial_s \partial_t \mu_1^s(x, 0) - \partial_s b_0 \partial_s \partial_t \mu_2^s(x, 0) - \partial_s \partial_t b_0 \partial_s \partial_t \mu_2^s(x, 0) - \varepsilon \partial_s \partial_t \mu_2^s \partial_s \partial_t \mu_2^s(x, 0). \\
\end{align*} \]

Likewise, we can also derive that \( \mu_2^s \) satisfies

\[ \begin{align*}
\partial_s^4 \mu_2^s &= -3(\partial_s \partial_t b_0 \partial_s \partial_t \mu_1^s + \partial_s \partial_t \mu_2^s \partial_s \partial_t \mu_1^s + \varepsilon \partial_s \partial_t \mu_1^s \partial_s \partial_t \mu_2^s) \\
&\quad + 3(\partial_s u_0 \partial_s \partial_t \mu_2^s + \partial_s \partial_t b_0 \partial_s \partial_t \mu_1^s + \varepsilon \partial_s \partial_t \mu_1^s \partial_s \partial_t \mu_2^s) + 3 \partial_s \partial_t u_0^s(0) \partial_s \partial_t \mu_2^s. \\
\end{align*} \]

Taking the values at \( t = 0 \) for (4.19), we attain a restraint condition for \( (\partial_s^4 \mu_1^s, \partial_s^4 \mu_2^s) \),

\[ \begin{align*}
\partial_s^4 \mu_1^s(x, 0) &= -2(\partial_s^3 \partial_t b_0 \partial_s \partial_t \mu_1^s + \partial_s^3 \partial_s \partial_t \mu_2^s \partial_s \partial_t \mu_1^s + \varepsilon \partial_s^3 \partial_s \partial_t \mu_2^s \partial_s \partial_t \mu_1^s)(x, 0) \\
&\quad + (\partial_s \partial_t u_0 \partial_s^3 \mu_1^s + \partial_s \partial_t \mu_2^s \partial_s^3 \mu_1^s + \varepsilon \partial_s \partial_t \mu_1^s \partial_s^3 \mu_1^s)(x, 0) \\
&\quad + \partial_s^3 \partial_t u^s \partial_s \partial_t \mu_1^s(x, 0) - 2 \partial_s^2 \partial_t \partial_s \partial_t \mu_2^s(x, 0) \\
&\quad - (\partial_s^3 \partial_t b_0 \partial_s \partial_t \mu_2^s + \partial_s^3 \partial_s \partial_t \mu_2^s \partial_s \partial_t \mu_2^s + \varepsilon \partial_s^3 \partial_s \partial_t \mu_2^s \partial_s \partial_t \mu_2^s)(x, 0) \\
&\quad + 2(\partial_s b_0 \partial_s^2 \partial_t \partial_s \partial_t \mu_1^s + \partial_s \partial_t b_0 \partial_s^2 \partial_s \partial_t \mu_1^s + \varepsilon \partial_s \partial_t \mu_1^s \partial_s^2 \partial_s \partial_t \mu_1^s)(x, 0) \\
&\quad + 2(\partial_s \partial_t b_0 \partial_s^2 \partial_s \partial_t \mu_1^s + \varepsilon \partial_s \partial_t \mu_2^s \partial_s^2 \partial_s \partial_t \mu_2^s + \varepsilon^2 \partial_s \partial_t \mu_1^s \partial_s^2 \partial_s \partial_t \mu_2^s)(x, 0) \\
&\quad - 2(\partial_s^2 \partial_t \partial_s \partial_t \mu_1^s + \varepsilon \partial_s \partial_t \partial_s \partial_t \mu_1^s + \varepsilon \partial_s \partial_t \mu_2^s \partial_s \partial_t \mu_1^s \partial_s \partial_t \mu_1^s + \varepsilon \partial_s \partial_t \mu_1^s \partial_s^2 \partial_s \partial_t \mu_2^s)(x, 0) \\
&\quad + \sum_{1 \leq j \leq 3} C_j^1(\partial_s^4 (u^s + u_0) \partial_s \partial_t \mu_1^s - \partial_s \partial_t \mu_2^s \partial_s \partial_t \mu_1^s + \varepsilon \partial_s \partial_t \mu_1^s \partial_s \partial_t \mu_2^s)(x, 0) \\
&\quad - \sum_{1 \leq j \leq 3} C_j^1(\partial_s \partial_t \mu_2^s \partial_s \partial_t \mu_1^s + \partial_s \partial_t \mu_1^s \partial_s \partial_t \mu_2^s + \varepsilon \partial_s \partial_t \mu_1^s \partial_s \partial_t \mu_2^s)(x, 0) \\
&\quad - \sum_{1 \leq j \leq 3} C_j^1(\partial_s \partial_t \mu_2^s \partial_s \partial_t \mu_1^s + \partial_s \partial_t \mu_1^s \partial_s \partial_t \mu_2^s + \varepsilon \partial_s \partial_t \mu_1^s \partial_s \partial_t \mu_2^s)(x, 0) \\
&\quad + \sum_{1 \leq j \leq 3} C_j^1(\partial_s \partial_t \mu_2^s \partial_s \partial_t \mu_1^s + \partial_s \partial_t \mu_1^s \partial_s \partial_t \mu_2^s + \varepsilon \partial_s \partial_t \mu_1^s \partial_s \partial_t \mu_2^s)(x, 0) \\
&\quad - \sum_{1 \leq j \leq 3} C_j^1 \partial_s \partial_t \mu_2^s(0) \partial_s \partial_t \mu_1^s(x, 0).
\end{align*} \]

Similarly as above

\[ \begin{align*}
\partial_s^4 \mu_2^s &= -3(\partial_s^3 \partial_t b_0 \partial_s \partial_t \mu_1^s + \partial_s^3 \partial_s \partial_t \mu_2^s \partial_s \partial_t \mu_1^s + \varepsilon \partial_s^3 \partial_s \partial_t \mu_2^s \partial_s \partial_t \mu_1^s)(x, 0) \\
&\quad + 4(\partial_s \partial_t b_0 \partial_s^3 \mu_1^s + \varepsilon \partial_s \partial_t \mu_1^s \partial_s^3 \mu_1^s)(x, 0) \\
&\quad - 4(\partial_s b_0 \partial_s^2 \partial_t \partial_s \partial_t \mu_1^s + \partial_s \partial_t \mu_2^s \partial_s^2 \partial_s \partial_t \mu_1^s + \varepsilon \partial_s \partial_t \mu_1^s \partial_s^2 \partial_s \partial_t \mu_2^s)(x, 0) \\
&\quad - 2(\partial_s b_0 \partial_s^2 \partial_t \partial_s \partial_t \mu_1^s + \partial_s \partial_t \mu_2^s \partial_s^2 \partial_s \partial_t \mu_1^s + \varepsilon \partial_s \partial_t \mu_1^s \partial_s^2 \partial_s \partial_t \mu_2^s)(x, 0) \\
&\quad + 4(\partial_s \partial_t b_0 \partial_s \partial_t \mu_1^s + \partial_s \partial_t \mu_2^s \partial_s \partial_t \mu_1^s + \varepsilon \partial_s \partial_t \mu_1^s \partial_s \partial_t \mu_2^s)(x, 0) \\
&\quad + 2(\partial_s \partial_t \mu_1^s \partial_s \partial_t \mu_1^s + \partial_s \partial_t \mu_1^s \partial_s \partial_t \mu_1^s + \varepsilon \partial_s \partial_t \mu_1^s \partial_s \partial_t \mu_1^s)(x, 0)
\end{align*} \]
which are not equal to zero, respectively. Thus, it is necessary to add the corrector terms and those of initial data (underlined terms in (4.18) and (4.21), respectively. All these underlined terms are from the added regular on the boundary. This explain also why we add corrector terms for the initial data in (4.1).

We now construct the polynomial functions \( \tilde{\mu}_1(x, y) \) and \( \tilde{\mu}_2(x, y) \) on \( y \) by the following forms

\[
\tilde{\mu}_1(x, y) = \mu_1^0(x) \frac{y^6}{6!} \quad \text{and} \quad \tilde{\mu}_2(x, y) = \mu_2^0(x) \frac{y^6}{6!},
\]

where

\[
\mu_1^0(x) = 2(\partial_1 \partial_3 b_0 \partial_2^3 \partial_0 - \partial_2^3 \partial_1 \partial_0 \partial_3 u_0) \quad \text{and} \quad \mu_2^0(x) = 6(\partial_1 \partial_2 b_0 \partial_3 \partial_0 - \partial_2^3 \partial_1 \partial_0 \partial_3 u_0).
\]

We take \( \mu_1^0(x, y) = k(y)\tilde{\mu}_1(x, y) \) and \( \mu_2^0(x, y) = k(y)\tilde{\mu}_2(x, y) \) with \( k \in C^\infty(\mathbb{R}_+); k(y) = 1, y \in [0, 1]; k(y) = 0, y \geq 2 \). Thus the proof is thus completed.

\[\Box\]

**Remark 4.1.** Actually, if we take \( (\mu_1^0(x, y), \mu_2^0(x, y)) \) with

\[
\partial_1^j \mu_1^0(x, 0) = 0 \quad \text{and} \quad \partial_1^j \mu_2^0(x, 0) = 0, \quad 0 \leq j \leq 5.
\]

Then (4.27) and (4.28) imply

\[
\begin{cases}
\partial_1^5 \mu_1^0(x, 0) = -2(\partial_1^2 \partial_0 \partial_3 \partial_0 \partial_3 u_0 + 2\partial_3 b_0 \partial_2^3 \partial_0 - \partial_2^3 \partial_1 \partial_0 \partial_3 u_0), \\
\partial_1^5 \mu_2^0(x, 0) = 6(\partial_1 \partial_2 b_0 \partial_3 \partial_0 - \partial_2^3 \partial_1 \partial_0 \partial_3 u_0),
\end{cases}
\]

which are not equal to zero, respectively. Thus, it is necessary to add the corrector terms \( \mu_1^0 \) and \( \mu_2^0 \) for the initial data of the regularized system, respectively.
5. The approximate solutions of the MHD boundary layer equations

In this section, we prove the existence of approximate solutions by establishing a series of the estimates of solutions for the nonlinear MHD boundary layer problem (4.1). To be more specifically, we plan to complete the proof of the solution for problem (4.1) by the following two subsections. In the first subsection, we will attain the weighted estimates for $D^\alpha(u, b)$ for $\alpha = (\beta, k) = (\beta_1, \beta_2, k)$ satisfying $\alpha = |\beta| + k \leq 4$ and $|\beta| \leq 3$. And the weighted estimates for $D^\beta(u, h)$ with $|\beta| = 4$ in the second subsection.

Firstly, we introduce the following lemma, which is helpful to deal with the boundary value.

**Lemma 5.1.** (11) Let $1 < p < \infty$. If $U \in W^{m,p}(\mathbb{R}^{n+1})$, then its trace $u$ belongs to the space $B = B^{m-1,p}_p(\mathbb{R}^n)$ and

$$\|u\|_B \leq K\|U\|_{W^{m,p}(\mathbb{R}^{n+1})},$$

with the constant $K > 0$ independent of $U$.

**Corollary 5.1.** Let $1 < p < \infty$. If $U \in W^{m,p}(\mathbb{R}^{n+1})$, then its trace $u$ belongs to the space $W^{m-1,p}(\mathbb{R}^n)$ and

$$\|u\|_{W^{m-1,p}(\mathbb{R}^n)} \leq K\|U\|_{W^{m,p}(\mathbb{R}^{n+1})},$$

with the constant $K > 0$ independent of $U$.

**Proof.** Since $1 < p < \infty$, it follows from the fact $W^{m-1,p}(\mathbb{R}^n) = F^{m-1,p}_p(\mathbb{R}^n)$ and the embedding theorem $B^{m-1-p}_p(\mathbb{R}^n) \hookrightarrow F^{m-1,p}_p(\mathbb{R}^n)$ in [24] that $B^{m-1-p}_p(\mathbb{R}^n) \hookrightarrow B^{m-1}_p(\mathbb{R}^n) \hookrightarrow W^{m-1,p}(\mathbb{R}^n)$, which gives

$$\|u\|_{W^{m-1,p}(\mathbb{R}^n)} \leq C\|U\|_{B^{m-p,p}_{p,p}(\mathbb{R}^n)},$$

which, together with Lemma 5.1, completes the proof.

5.1. Weighted $H^l_{k+1}$ with normal derivatives

We use energy methods to establish the weighted estimates for $D^\alpha(u, b)$ with $\alpha = (\beta, k) = (\beta_1, \beta_2, k)$ and $\alpha = |\beta| + k \leq 4$ and $|\beta| \leq 3$. That is, we have the following lemma:

**Lemma 5.2.** Let $k \geq \frac{1}{2}$, $l \geq 0$ be real numbers. Assume that $(u^\varepsilon, v^\varepsilon, g^\varepsilon, b^\varepsilon)$ is a solution to the problem (4.1) in $[0, T]$ and satisfies $(u^\varepsilon, b^\varepsilon) \in L^\infty([0, T]; H^l_{k+1})$, then there exists a positive constant $C$, may be depend on $k, l$ such that

$$\sum_{|\alpha| \leq 4, \beta \leq 3} \left( \frac{d}{dt} \|D^\alpha(u^\varepsilon, b^\varepsilon)(t)\|_{L^2_{k+1}}^2 + \varepsilon\|D^\alpha \partial_\alpha(u^\varepsilon, b^\varepsilon)(t)\|_{L^2_{k+1}}^2 + \|D^\alpha \partial_\alpha(u^\varepsilon, b^\varepsilon)(t)\|_{L^2_{k+1}}^2 \right) \leq C\|(u^\varepsilon, b^\varepsilon)\|_{H^l_{k+1}}^2 + C\|(u^\varepsilon, b^\varepsilon)\|_{H^l_{k+1}}^2. \quad (5.1)$$

**Proof.** Applying the operator $D^\alpha = \partial^\alpha_{i_1} \cdots \partial^\alpha_{i_l}$ on (4.1)$_{1,2}$ for $\alpha = (\beta, k) = (\beta_1, \beta_2, k)$ satisfying $\alpha = |\beta| + k \leq 4$ and $|\beta| \leq 3$, we have

$$\begin{cases}
\partial_t D^\alpha u^\varepsilon - \varepsilon \partial^\alpha_{i_1} D_{i_1} u^\varepsilon = \partial^2_{i_1} D^\alpha u^\varepsilon - D^\alpha((u^s + u^e) \partial_i u^e + v^e \partial_i u^e) \\
\quad - (1 + b^e) \partial_i \partial_j b^e - g^e \partial_i b^e + v^e \partial_i u^e \\
\partial_t D^\beta b^\varepsilon - \varepsilon \partial^\beta_{i_1} D_{i_1} b^\varepsilon = \partial^2_{i_1} D^\beta b^\varepsilon - D^\beta((u^s + u^e) \partial_i b^e + v^e \partial_i b^e) \\
\quad - (1 + b^e) \partial_i u^e - g^e \partial_i u^e - g^e \partial_i u^e. \quad (5.2)
\end{cases}$$
Multiplying the resulting equation (5.2)₁₂ by \( \langle y \rangle^{2k+2l}D^\alpha u^\varepsilon \) and \( \langle y \rangle^{2k+2l}D^\alpha b^\varepsilon \) respectively, and integrating it by parts over \( \mathbb{R}^2 \), we derive that

\[
\frac{1}{2} \frac{d}{dt} \| \langle y \rangle^{k+1} D^\alpha (u^\varepsilon, b^\varepsilon) (t) \|_{L^2}^2 + \varepsilon \| \langle y \rangle^{k+1} D^\alpha \partial_x (u^\varepsilon, b^\varepsilon) (t) \|_{L^2}^2
= \int_{\mathbb{R}^2} r_1 \cdot \langle y \rangle^{2k+2l} D^\alpha u^\varepsilon \, dx \, dy + \int_{\mathbb{R}^2} r_2 \cdot \langle y \rangle^{2k+2l} D^\alpha b^\varepsilon \, dx \, dy 
+ \int_{\mathbb{R}^2} \partial_x^2 D^\alpha u^\varepsilon \cdot \langle y \rangle^{2k+2l} D^\alpha u^\varepsilon \, dx \, dy + \int_{\mathbb{R}^2} \partial_x^2 D^\alpha b^\varepsilon \cdot \langle y \rangle^{2k+2l} D^\alpha b^\varepsilon \, dx \, dy,
\]

where

\[
\begin{cases} 
  r_1 = -D^\alpha ((u^\varepsilon + u^\varepsilon) \partial_x D^\alpha u^\varepsilon + \varepsilon \partial_x D^\alpha u^\varepsilon - (1 + b^\varepsilon) \partial_x b^\varepsilon - g^\varepsilon \partial_x b^\varepsilon + \varepsilon \partial_x u^\varepsilon) + D^\alpha (\varepsilon \partial_x u^\varepsilon) \equiv r_1 + r_2 + r_3, \\
  r_2 = -D^\alpha ((u^\varepsilon + u^\varepsilon) \partial_x D^\alpha b^\varepsilon + \varepsilon \partial_x D^\alpha b^\varepsilon - (1 + b^\varepsilon) \partial_x b^\varepsilon - g^\varepsilon \partial_x D^\alpha u^\varepsilon) - (D^\alpha, (u^\varepsilon + u^\varepsilon)) \partial_x b^\varepsilon + [D^\alpha, \varepsilon] \partial_x b^\varepsilon - [D^\alpha, (1 + b^\varepsilon)] \partial_x u^\varepsilon - [D^\alpha, g^\varepsilon] \partial_x u^\varepsilon - D^\alpha (g^\varepsilon \partial_x u^\varepsilon) \equiv r_1 + r_2 + r_3.
\end{cases}
\]

Next, we will establish the estimates of the right-hand side nonlinear terms (5.3). First of all, we deal with the first second terms. By the definitions of \( r_1 \) and \( r_2 \), we have

\[
\begin{align*}
  r_1 &= \int_{\mathbb{R}^2} r_1 \cdot \langle y \rangle^{2k+2l} D^\alpha u^\varepsilon \, dx \, dy + \int_{\mathbb{R}^2} r_2 \cdot \langle y \rangle^{2k+2l} D^\alpha b^\varepsilon \, dx \, dy,
  \\
  &\text{into the following three parts:}
  \\
  &\int_{\mathbb{R}^2} r_1 \cdot \langle y \rangle^{2k+2l} D^\alpha u^\varepsilon \, dx \, dy + \int_{\mathbb{R}^2} r_2 \cdot \langle y \rangle^{2k+2l} D^\alpha b^\varepsilon \, dx \, dy
  \\
  &= \sum_{i=1}^3 \int_{\mathbb{R}^2} r_i^2 \cdot \langle y \rangle^{2k+2l} D^\alpha u^\varepsilon \, dx \, dy + \int_{\mathbb{R}^2} r_i^2 \cdot \langle y \rangle^{2k+2l} D^\alpha b^\varepsilon \, dx \, dy \quad \text{(5.4)}
  \\
  &\equiv I_1 + I_2 + I_3,
\end{align*}
\]

and the estimates of each term \( I_i \) as follows.

\textbf{The estimate for} \( I_1 \):

\[
\int_{\mathbb{R}^2} r_i^1 \cdot \langle y \rangle^{2k+2l} D^\alpha u^\varepsilon \, dx \, dy + \int_{\mathbb{R}^2} r_i^1 \cdot \langle y \rangle^{2k+2l} D^\alpha b^\varepsilon \, dx \, dy
\]
\[ = 2(k + 1) \int_{\mathbb{R}^2} \langle y \rangle^{2k+2l-1}(v^\varepsilon, g^\varepsilon)(D^a u^\varepsilon)^2 dxdy + |D^a b^\varepsilon|^2 dxdy \tag{5.5} \]
\[ \leq C\|\langle v^\varepsilon, g^\varepsilon\rangle\|_{L^1} \|\langle y \rangle^{2k+2l-1} D^a (u^\varepsilon, b^\varepsilon)\|^2_{L^2} \]
\[ \leq C\|\langle u^\varepsilon, b^\varepsilon\rangle\|_{H^0_{k+1}} \|\langle u^\varepsilon, b^\varepsilon\rangle\|^2_{H^m_{k+1}}, \]

where we have used the integration by parts.

**The estimate for** \( I_2 \):

Noticing that

\[ I_2 \leq \|r_2^2\|_{L^2_{k+1}} \|D^a u^\varepsilon\|_{L^2_{k+1}^2} + \|r_2^2\|_{L^2_{k+1}} \|D^a b^\varepsilon\|_{L^2_{k+1}^2}, \tag{5.6} \]

Therefore, we need to establish the estimates of the terms \( \|r_2^2\|_{L^2_{k+1}} \) and \( \|r_2^2\|_{L^2_{k+1}^2} \). However, we know that the terms in \( \|r_2^2\|_{L^2_{k+1}} \) are similar to the terms in \( \|r_1^2\|_{L^2_{k+1}} \), we will estimate only the \( L^2_{k+1} \) of \( r_1^2 \).

For the commutator operator, we can rewrite it as

\[ [D^a, (u^\varepsilon + u^\varepsilon)]\partial_j u^\varepsilon = \sum_{\hat{a} \lesssim \alpha, \kappa \lesssim \hat{a}} C^\varepsilon_{\alpha} \partial^\varepsilon (u^\varepsilon + u^\varepsilon) \partial^{\alpha - \hat{a}} \partial_j u^\varepsilon. \]

Then for \( \alpha \leq 4 \), we can obtain

\[ \|[D^a, (u^\varepsilon + u^\varepsilon)]\partial_j u^\varepsilon\|_{L^2_{k+1}} \leq C(\|u^\varepsilon\|_{H^4_{k+1}} + \|u^\varepsilon\|^2_{H^4_{k+1}}). \]

Noting that

\[ [D^a, v^\varepsilon]\partial_j u^\varepsilon = \sum_{\hat{a} \lesssim \alpha, \kappa \lesssim \hat{a}} C^\varepsilon_{\alpha} \partial^\varepsilon v^\varepsilon \partial^{\alpha - \hat{a}} \partial_j u^\varepsilon. \]

Since \( 1 \leq |\hat{a}|, \hat{a} \leq \alpha \), we have for \( |\alpha - \hat{a}| \leq 3 \),

\[ -\partial^\varepsilon v^\varepsilon = \partial^\varepsilon \partial^\varepsilon \int_0^y \partial_j u^\varepsilon d\tilde{y} = \left\{ \begin{array}{ll}
\partial^\varepsilon \partial^\varepsilon \partial^\varepsilon \partial^\varepsilon u^\varepsilon, & k \geq 1, \\
\int_0^y \partial^\varepsilon \partial^\varepsilon u^\varepsilon d\tilde{y}, & k = 0.
\end{array} \right. \]

Thus, using Lemma 2.2, we can arrive at

\[ \|[\partial^\varepsilon \partial^\varepsilon \partial^\varepsilon \partial^\varepsilon u^\varepsilon \partial^{\alpha - \hat{a}} \partial_j u^\varepsilon (t, \cdot)]\|_{L^2_{k+1}} \leq C\|u^\varepsilon\|_{H^4_{k+1}} \|u^\varepsilon\|_{H^0_{k+1}} \tag{5.7} \]

and

\[ \|[\partial^\varepsilon \partial^\varepsilon u^\varepsilon \partial^{\alpha - \hat{a}} \partial_j u^\varepsilon (t, \cdot)]\|_{L^2_{k+1}} \leq C\|u^\varepsilon\|_{H^4_{k+1}} \|u^\varepsilon\|_{H^0_{k+1}}. \tag{5.8} \]

Combining (5.7) with (5.8), we derive

\[ \|[D^a, v^\varepsilon]\partial_j u^\varepsilon\|_{L^2_{k+1}} \leq C\|u^\varepsilon\|^2_{H^4_{k+1}}. \]

Similarly, we also have

\[ \|[D^a, (1 + b^\varepsilon)]\partial_j b^\varepsilon\|_{L^2_{k+1}} \leq C(\|b^\varepsilon\|_{H^4_{k+1}} + \|b^\varepsilon\|^2_{H^4_{k+1}}). \]
and
\[ \|([D^a, g^e] \partial_x g^f)\|_{L^2_{x,t}} \leq C \|b^e\|_{H^k_{x,t}}. \]

Using the above three inequalities, we can attain
\[ \|r_2^2\|_{L^2_{x,t}} \leq C(\|(u^e, b^f)\|_{H^4_{x,t}} + \|(u^e, b^f)\|_{H^4_{x,t}}^2). \]

(5.9)

Similarly as above for \( L^2_{k+1} \) of \( r_2^2 \), we conclude that
\[ \|r_2^2\|_{L^2_{x,t}} \leq C(\|(u^e, b^f)\|_{H^4_{x,t}} + \|(u^e, b^f)\|_{H^4_{x,t}}^2). \]

(5.10)

Inserting (5.9) and (5.10) into (5.6), we have
\[ I_2 \leq C(\|(u^e, b^f)\|_{H^4_{x,t}} + \|(u^e, b^f)\|_{H^4_{x,t}}^2). \]

(5.11)

**The estimate for \( I_3 \):**
By a direct computation, we can get for \(|\beta| \geq 1, k + |\beta| \leq 4,\)
\[
\sum_{|\alpha| = |\beta| + |\gamma| \leq 1, k + |\gamma| \leq 4} \partial_\gamma^\beta \partial_\gamma^\beta \partial_\gamma^\beta\langle y u^\gamma \rangle
\]
\[
= \sum_{|\alpha| \leq 4, \sum k \geq 1, \sum l \leq \sum \beta + \sum \gamma \leq 3} C_k \epsilon_\beta \epsilon_\gamma \epsilon_\gamma \partial_\alpha^\beta \partial_\alpha^\beta \partial_\alpha^\beta y u^\gamma
\]
\[
= \sum_{|\alpha| \leq 4, \sum k \geq 1, \sum l \leq \sum \beta + \sum \gamma \leq 3} C_k \epsilon_\beta \epsilon_\gamma \epsilon_\gamma \partial_\alpha^\beta \partial_\alpha^\beta \partial_\alpha^\beta y u^\gamma.
\]

Thus, for \(|\alpha| \leq 4, \) we can get
\[ \|\partial_\gamma^\beta (y u^\gamma)\|_{L^2_{x,t}} \leq C \|u^\gamma\|_{H^4_{x,t}}. \]

Analogously, for \(|\alpha| \leq 4, \) we also can derive
\[ \|\partial_\gamma^\beta (g^e u^\gamma)\|_{L^2_{x,t}} \leq C \|b^e\|_{H^4_{x,t}}. \]

Combining the above two inequalities with \(|\alpha| \leq 4, \) we have
\[ I_3 \leq C(\|(u^e, b^f)\|_{H^4_{x,t}}^2). \]

(5.12)

Hence, we infer from (5.5), (5.11) and (5.12),
\[
\left| \int_{\mathbb{R}^2_x} r_1 \cdot \langle y \rangle^{2k+2l} D^a u^e dxdy + r_2 \cdot \langle y \rangle^{2k+2l} D^a b^f dxdy \right| \leq C(\|(u^e, b^f)\|_{H^4_{x,t}}^2 + \|(u^e, b^f)\|_{H^4_{x,t}}^4). \]

(5.13)

In the following part, we will estimate the remainder terms, we first deal with the term
\[ \int_{\mathbb{R}^2_x} \langle y \rangle^{2k+2l} \partial_\gamma^2 \partial_\gamma^2 D^a u^e \cdot D^a u^e dxdy. \] Similarly, the \[ \int_{\mathbb{R}^2_x} \langle y \rangle^{2k+2l} \partial_\gamma^2 \partial_\gamma^2 D^a b^f \cdot D^a b^f dxdy \] can be derived. Integrating it by parts and using the boundary value (4.2), we arrive at
\[
\int_{\mathbb{R}^2_x} \langle y \rangle^{2k+2l} \partial_\gamma^2 D^a u^e \cdot D^a u^e dxdy
\]

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\[ = -\|y\|_2^{2(k+l)} \partial_x D^a u^e \|_2^2 - 2(k+l) \int_{\mathbb{R}_+^2} \langle y \rangle^{2k+2l-1} \partial_x D^a u^e \cdot D^a u^e \, dx dy + \int_{\mathbb{R}} \partial_x D^a u^e \cdot D^a u^e \big|_{y=0} dx. \]  

(5.14)

Applying the Cauchy-Schwarz inequality, we can get

\[ 2(k+l) \int_{\mathbb{R}_+^2} \langle y \rangle^{2k+2l-1} \partial_x D^a u^e \cdot D^a u^e \, dx dy \leq \frac{1}{8} \|\partial_x u^e\|_{H^{4+1}_{x+1}}^4 + C \|u^e\|_{H^{4+1}_{x+1}}^2. \]  

(5.15)

Now, we study the last term in (5.14), that is the boundary integral \( \int_{\mathbb{R}} \partial_x D^a u^e \cdot D^a u^e \big|_{y=0} dx \). By a direct calculation, we know the boundary integral \( \int_{\mathbb{R}} \partial_x D^a u^e \cdot D^a u^e \big|_{y=0} dx = 0 \) when the cases \( k = 1, 2 \). Therefore, we only consider the cases \( k = 3, 4 \).

Case 1: \( |\beta| = 1, k = 3 \), using Corollary 5.1 and the boundary conditions \( y = 0 \), we lead to

\[ \left| \int_{\mathbb{R}} \partial_x \partial_{x}^3 u^e \cdot \partial_x \partial_{x}^3 u^e \big|_{y=0} dx \right| \leq \\|\partial_x \partial_{x}^3 u^e\|_{H^{4+1}_{x+1}}^2 + C \|u^e\|_{H^{4+1}_{x+1}}^4. \]  

(5.16)

Case 2: \( \beta_1 = \beta_2 = 0, k = 4 \), i.e., \( \left| \int_{\mathbb{R}} \partial_x \partial_{x}^3 u^e \cdot \partial_x \partial_{x}^3 u^e \big|_{y=0} dx \right| \). The estimate of this term is the main obstacle, since there is a higher order partial derivation in \( y \) on the boundary value, we use the equation (4.1), Corollary 5.1 and the boundary conditions to overcome this difficulty. We first get the boundary value of \( \partial_x \partial_{x}^3 u^e \big|_{y=0} \) by using the equation (4.1),

\[ \partial_x \partial_{x}^3 u^e = \partial_x \big( \partial_x u^e - e 2 \partial_x^2 u^e + (u^e + u^e) \partial_x u^e + v^e \partial_x u^e - (1 + b^e) \partial_x b^e - g^e \partial_x b^e + v^e \partial_x u^e \big). \]

Then, using the boundary value (4.2), we can obtain

\[ \left| \int_{\mathbb{R}} \partial_x \partial_{x}^3 u^e \cdot \partial_x \partial_{x}^3 u^e \big|_{y=0} dx \right| \]

\[ = \left| \int_{\mathbb{R}} \partial_x \big( \partial_x u^e - e 2 \partial_x^2 u^e + (u^e + u^e) \partial_x u^e + v^e \partial_x u^e - (1 + b^e) \partial_x b^e - g^e \partial_x b^e + v^e \partial_x u^e \big) \times \big( \partial_x \partial_x^3 u^e - \partial_x \partial_x^3 u^e \big) \big|_{y=0} dx \right| \]

\[ = \left| \int_{\mathbb{R}} \big( \partial_x \partial_x^3 u^e - e 2 \partial_x^2 \partial_x^3 u^e + (u^e + u^e) \partial_x \partial_x^3 u^e - \partial_x \partial_x^3 b^e \big) \times \big( \partial_x \partial_x^3 u^e - \partial_x \partial_x^3 u^e \big) \big|_{y=0} dx \right|. \]  

(5.17)

From the above equality, we only establish the estimate of the term which contains the term \( \partial_x^2 \partial_x^3 u^e \) by performing an integration by parts in \( x \) and using Corollary 5.1, which is the main difficulty. Its estimate show as follows

\[ - \int_{\mathbb{R}} \partial_x^2 \partial_x^3 u^e \big( \partial_x \partial_x^3 u^e \partial_x u^e + \partial_x \partial_x^3 u^e \partial_x u^e - \partial_x \partial_x^3 b^e \big) \big|_{y=0} dx \]

\[ \leq \frac{1}{8} \|\partial_x \partial_x^3 u^e\|_{H_{x+1}^{4+1}}^2 + C \|u^e\|_{H_{x+1}^{4+1}}^4. \]
\[= \int_R \partial_t \partial_x^3 u^e \partial_x^2 \left( \partial_x \partial_t u^e \partial_x u^e + \partial_x \partial_t u^e \partial_x u^e - \partial_x \partial_t u^e \partial_x u^e \right) \big|_{y=0} \, dx\]
\[\leq \left\| \partial_t \partial_x^3 u^e \right\|_{y=0} \left\| \partial_x \partial_t u^e \partial_x u^e + \partial_x \partial_t u^e \partial_x u^e - \partial_x \partial_t u^e \partial_x u^e \right\|_{y=0}\]
\[\leq \left\| \partial_t \partial_x^3 u^e \right\| \left\| \partial_x \partial_t u^e \partial_x u^e + \partial_x \partial_t u^e \partial_x u^e - \partial_x \partial_t u^e \partial_x u^e \right\|_{y=0}\]
\[\leq \frac{1}{32} \left\| \partial_t u^e \right\|_{H_{x,t}^4}^2 + C \left\| u^e \right\|_{H_{x,t}^4}^2 + C \left\| (u^e, b^e) \right\|_{H_{x,t}^4}^4. \quad (5.18)\]

Other terms are a direct calculations in (5.17) by using the Hölder, Young’s inequalities and Corollary 5.1. Hence, we deduce
\[\left\| \int_R \partial_t \partial_x^3 u^e \cdot \partial_x \partial_t u^e \right\|_{y=0} \right| \leq \frac{1}{8} \left\| \partial_t u^e \right\|_{H_{x,t}^4}^2 + C \left\| (u^e, b^e) \right\|_{H_{x,t}^4}^2 + C \left\| (u^e, b^e) \right\|_{H_{x,t}^4}^4. \quad (5.19)\]

Inserting (5.15), (5.16) and (5.19) into (5.14), we arrive at
\[\left\| \int \partial_t^2 D^a u^e \cdot \langle y \rangle^{2k+2} D^a u^e \, dxdy \right\| \leq - \frac{5}{8} \left\| \partial_t u^e \right\|_{H_{x,t}^4}^2 + C \left\| (u^e, b^e) \right\|_{H_{x,t}^4}^2 + C \left\| (u^e, b^e) \right\|_{H_{x,t}^4}^4. \quad (5.20)\]

Similar to (5.20), we easily conclude that
\[\left\| \int \partial_t^2 D^a b^e \cdot \langle y \rangle^{2k+2} D^a b^e \, dxdy \right\| \leq - \frac{5}{8} \left\| \partial_t b^e \right\|_{H_{x,t}^4}^2 + C \left\| (u^e, b^e) \right\|_{H_{x,t}^4}^2 + C \left\| (u^e, b^e) \right\|_{H_{x,t}^4}^4. \quad (5.21)\]

Plugging (5.13), (5.20) and (5.21) into (5.5), yields (5.1). The proof is thus completed. \(\square\)

**Lemma 5.3.** Let \( k \geq \frac{1}{2} \), \( l \geq 0 \) be real numbers. Assume that \( (u^e, v^e, g^e, b^e) \) is a solution to the problem (4.1) in \([0, T]\) and satisfies \( (u^e, b^e) \in L^\infty ([0, T], H_{x,t}^1) \), then there exists a positive constant \( C \), may be depend on \( k, l \) such that
\[\sum_{|\beta|=4} \left( \frac{d}{dt} \left| \partial_t^2 (u^e, b^e) (t) \right|_{L^2_{x,t}}^2 + \varepsilon \left| \partial_t^2 \partial_x (u^e, b^e) (t) \right|_{L^2_{x,t}}^2 + \left| \partial_t \partial_x (u^e, b^e) (t) \right|_{L^2_{x,t}}^2 \right) \leq C \left| (u^e, b^e) \right|_{H_{x,t}^4}^2 + C \left| (u^e, b^e) \right|_{H_{x,t}^4}^4 + \frac{C}{\varepsilon} \left| (u^e, b^e) \right|_{H_{x,t}^4}^2. \quad (5.22)\]

**Proof.** Applying the operator \( \partial_t^2 = \partial_t^2 \partial_x^2 \) on (4.1)$_{1,2}$ and \(||\beta|| = ||\beta_1|| + ||\beta_2|| = 4,$
\[\begin{align*}
\partial_t \partial_x^2 u^e - \varepsilon \partial_t \partial_x^2 b^e &= \partial_t^2 \partial_x^2 u^e - \partial_t^2 \left( (u^e + u^e) \partial_x u^e + v^e \partial_x u^e \right)
- (1 + b^e) \partial_x b^e - g^e \partial_x b^e + v^e \partial_x u^e, \\
\partial_t \partial_x^2 b^e - \varepsilon \partial_t \partial_x^2 b^e &= \partial_t^2 \partial_x^2 b^e - \partial_t^2 \left( (u^e + u^e) \partial_x b^e + v^e \partial_x b^e \right)
- (1 + b^e) \partial_x u^e - g^e \partial_x u^e + v^e \partial_x b^e.
\end{align*}\quad (5.23)\]

Multiplying the resulting equation (5.23)$_{1,2}$ by \( \langle y \rangle^{2k+2} \partial_t^2 u^e \) and \( \langle y \rangle^{2k+2} \partial_t^2 b^e \) respectively, and integrating it by parts over \( \mathbb{R}^2 \), we derive that
\[\frac{1}{2} \frac{d}{dt} \left| \langle y \rangle^{k+l} \partial_t^2 (u^e, b^e) (t) \right|_{L^2_{x,t}}^2 + \varepsilon \left| \langle y \rangle^{k+l} \partial_t \partial_x (u^e, b^e) (t) \right|_{L^2_{x,t}}^2 \]
and the estimates of each term $J_i$ are as following. 

The estimate for $J_1$: 

$$
\int_{\mathbb{R}^2} R_1 \cdot \langle y \rangle^{2k+2l} \partial^\beta_x u^d \, dx dy
= 2(k + l) \int_{\mathbb{R}^2} \langle y \rangle^{2k+2l-1} (v^e, g^e)(|\partial^\beta_x u^d|^2 \, dx dy + |\partial^\beta_y b^e|^2) \, dx dy
\leq C \|(v^e, g^e)\|_{L^2} \|\langle y \rangle^{2k+2l-1} \partial^\beta_x (u^d, b^e)\|^2_{L^4_{x,y}}.
$$

Next, we will establish the estimates of right-hand side terms (5.24). First of all, we deal with the first second terms. By the definitions of $R_1$ and $R_2$, we have 

$$
\begin{align*}
R_1 &= -\partial^\beta_x ((u^e + u^f) \partial_x u^e + v^e \partial_y u^e - (1 + b^f) \partial_x b^f - g^e \partial_y b^f + v^e \partial_y u^e), \\
R_2 &= -\partial^\beta_x ((u^e + u^f) \partial_x b^e + v^e \partial_y b^e - (1 + b^f) \partial_x b^e - g^e \partial_y b^e).
\end{align*}
$$

Hence, we can divide the term 

$$
\int_{\mathbb{R}^2} R_1 \cdot \langle y \rangle^{2k+2l} \partial^\beta_x u^d \, dx dy + R_2 \cdot \langle y \rangle^{2k+2l} \partial^\beta_y b^e \, dx dy
$$

into the following three parts: 

$$
\int_{\mathbb{R}^2} R_1 \cdot \langle y \rangle^{2k+2l} \partial^\beta_x u^d \, dx dy + R_2 \cdot \langle y \rangle^{2k+2l} \partial^\beta_y b^e \, dx dy
= \sum_{i=1}^{3} \int_{\mathbb{R}^2} R_i \cdot \langle y \rangle^{2k+2l} \partial^\beta_x u^d \, dx dy + R_i \cdot \langle y \rangle^{2k+2l} \partial^\beta_y b^e \, dx dy
= J_1 + J_2 + J_3,
$$

where 

$$
\begin{align*}
R_1 &= -\partial^\beta_x ((u^e + u^f) \partial_x u^e + v^e \partial_y u^e - (1 + b^f) \partial_x b^f - g^e \partial_y b^f + v^e \partial_y u^e), \\
R_2 &= -\partial^\beta_x ((u^e + u^f) \partial_x b^e + v^e \partial_y b^e - (1 + b^f) \partial_x b^e - g^e \partial_y b^e).
\end{align*}
$$

are as following.
where we have used integration by parts.

**The estimate for** $J_2$:

Noticing that

$$J_2 \leq ||R^2_1||_{L^2_{k+l}} ||D^u u^e||_{L^2_{k+l}} + ||R^2_2||_{L^2_{k+l}} ||D^b b^e||_{L^2_{k+l}}. \quad (5.27)$$

Therefore, we need to establish the estimates of the terms $||R^2_1||_{L^2_{k+l}}$ and $||R^2_2||_{L^2_{k+l}}$. However, we know that the terms in $||R^2_2||_{L^2_{k+l}}$ are similar to the terms in $||R^2_1||_{L^2_{k+l}}$, we will estimate only the $L^2_{k+l}$ of $R^2_1$.

For the commutator operator, we can rewrite it as

$$[\partial^\beta_x, (u^e + u^f)] \partial_x u^e = \sum_{\beta \leq \beta, \ 1 \leq \beta} C^\beta_x \partial^\beta(u^e + u^f)\partial^{\beta-\beta} \partial_x u^e. \quad (5.28)$$

Then for $|\beta| = 4$, we can obtain

$$\|[\partial^\beta_x, (u^e + u^f)] \partial_x u^e\|_{L^2_{l+k}} \leq C||u^e||_{H^4_{l+k}} + \|u^e\|_{H^4_{l+k}}^2. \quad (5.29)$$

Noting that

$$[\partial^\beta_x, v^f] \partial^\beta_x u^e = \sum_{\beta \leq \beta, \ 1 \leq \beta} C^\beta_x \partial^\beta v^f \partial^{\beta-\beta} \partial_x u^e. \quad (5.29)$$

Since $1 \leq |\hat{\beta}|, \hat{\beta} \leq \beta$, we have for $|\beta - \hat{\beta}| \leq 3$,

$$-\hat{\partial}^{\beta+\epsilon} v^f = \hat{\partial}^{\beta} \int_0^\gamma \partial_x u^e d\tilde{y} = \int_0^\gamma \hat{\partial}^{\beta+\epsilon} u^e d\tilde{y}. \quad (5.29)$$

Thus, we have

$$\|\hat{\partial}^{\beta+\epsilon} u^e \|_{L^2_{l+k}} \leq C\|u^e\|_{H^4_{l+k}} \|u^e\|_{H^4_{l+k}}. \quad (5.30)$$

i.e.,

$$\|[[\partial^\beta_x, v^f]] \partial_x u^e\|_{L^2_{l+k}} \leq C\|u^e\|_{H^4_{l+k}}^2. \quad (5.30)$$

Similarly,

$$\|[[\partial^\beta_x, (1 + b^e)] \partial_x b^e]\|_{L^2_{l+k}} \leq C\|b^e\|_{H^4_{l+k}} + \|b^e\|_{H^4_{l+k}}^2. \quad (5.31)$$

and

$$\|[[\partial^\beta_x, g^e]] \partial_x g^e\|_{L^2_{l+k}} \leq C\|b^e\|_{H^4_{l+k}}^2. \quad (5.31)$$

Combining the above inequalities, we can attain

$$||R^2_1||_{L^2_{k+l}} \leq C\|(u^e, b^e)\|_{H^4_{k+l}} + \|(u^e, b^e)\|_{H^4_{k+l}}^2. \quad (5.32)$$

Similarly as the above estimates for $L^2_{k+l}$ of $R^2_1$, we can conclude that

$$||R^2_2||_{L^2_{k+l}} \leq C\|(u^e, b^e)\|_{H^4_{k+l}} + \|(u^e, b^e)\|_{H^4_{k+l}}^2. \quad (5.33)$$
Inserting (5.28) and (5.29) into (5.27), we have
\[
J_2 \leq C(||(u^e, b^e)||_{H^{k+1}_x}^2 + ||(u^e, b^e)||_{H^{k+1}_t}^2).
\]
(5.30)

The estimate for J3:

By a direct computation, we can attain that \(|\beta| = 4,
\[
\partial_i^\beta (v^r u^e_i') = \sum_{|\beta_i|=|\beta|+|\beta_i|=4} \partial_i^{\beta_i} \partial_x^{\beta - \beta_i} (v^r u^e_i')
= \sum_{|\beta_1|+|\beta_2|=4, \beta_1 \leq \beta_2} C_{\beta} C_{\beta_1} \partial_i^{\beta_1} \partial_x^{\beta_i} v^r \partial_i^{\beta_i} \partial_s u^e.
\]
Therefore, \(|\beta| = 4, \) we can get
\[
||\partial_i^\beta (v^r u^e_i')||_{L^2_{x,t}} \leq C ||\partial_i \partial_x^\beta u^e||_{L^2_{x,t}}.
\]

Analogously, \(|\beta| = 4, \) we can also derive
\[
||\partial_i^\beta (g^r u^e_i')||_{L^2_{x,t}} \leq C ||\partial_i \partial_x^\beta b^e||_{L^2_{x,t}}.
\]

Combining the above two inequalities with \(|\beta| = 4, \) we have
\[
J_3 \leq \frac{\epsilon}{2} ||\partial_i^\beta (u^e, b^e)||_{L^2_{x,t}}^2 + \frac{C}{\epsilon} ||(u^e, b^e)||_{H^2_{x,t}}^2.
\]
(5.31)

Hence, we infer from (5.26), (5.30) and (5.31),
\[
\left| \int_{\mathbb{R}^2_x} R_1 \cdot \langle y \rangle^{2k+2} \partial_i^\beta u^e \, dxdy + R_2 \cdot \langle y \rangle^{2k+2} \partial_i^\beta b^e \, dxdy \right|
\leq C(||(u^e, b^e)||_{L^2_{x,t}}^2 + ||(u^e, b^e)||_{H^2_{x,t}}^2).
\]
(5.32)

Now, we deal with the last two terms in (5.24), we first estimate the term \(\int_{\mathbb{R}^2_x} \langle y \rangle^{2k+2} \partial_t^\beta \partial_t u^e \cdot \partial_i^\beta u^e \, dxdy.\)

Similarly, the term \(\int_{\mathbb{R}^2_x} \langle y \rangle^{2k+2} \partial_t^\beta \partial_t b^e \cdot \partial_i^\beta b^e \, dxdy \) can be derived.

Integrating it by parts and using the boundary value (4.2) and the Cauchy-Schwarz inequality, we arrive at
\[
\int_{\mathbb{R}^2_x} \partial_t^\beta \partial_t^\beta u^e \cdot \langle y \rangle^{2k+2} \partial_t^\beta u^e \, dxdy
= -\langle (y) \rangle^{2(k+l)} \partial_j \partial_t^\beta u^e \|_{L^2}^2 - 2(k + l) \int_{\mathbb{R}^2_x} \partial_j \partial_t^\beta u^e \cdot \langle (y) \rangle^{2k+2} \partial_t^\beta u^e \, dxdy
- \langle (y) \rangle^{2(k+l)} \partial_j \partial_t^\beta u^e \|_{L^2}^2 + \frac{1}{2} \partial_j u^e \|_{H^2_{x,t}}^2 + C ||u^e||_{H^2_{x,t}}^2
\leq -\frac{1}{2} ||\partial_t^\beta \partial_t u^e||_{L^2_{x,t}}^2 + C ||u^e||_{H^2_{x,t}}^2.
\]
(5.33)

Similarly, we can deduce that
\[
\int_{\mathbb{R}^2_x} \partial_t^\beta \partial_t b^e \cdot \langle y \rangle^{2k+2} \partial_t^\beta b^e \, dxdy \leq -\frac{1}{2} ||\partial_t^\beta \partial_t b^e||_{L^2_{x,t}}^2 + C ||b^e||_{H^2_{x,t}}^2.
\]
(5.34)

Plugging (5.32)-(5.34) into (5.24), we can conclude the desired result. The proof is so completed. □
5.2. Weighted $H^1_{k+1}$ Only in Tangential Derivatives

To investigate the existence of solution to problem (4.1), we encounter some difficulties. Similar to the Prandtl equation, the difficulty of solving problem (4.1) in the Sobolev framework is the loss of $x$-derivative in the terms $\nabla \cdot \mathbf{u}^\varepsilon - g^\varepsilon \partial_y \mathbf{b}^\varepsilon$ and $\nabla \cdot \mathbf{b}^\varepsilon - g^\varepsilon \partial_y \mathbf{u}^\varepsilon$ in the first and second equations of (4.1), respectively. In other words, $\nabla \cdot \mathbf{u}^\varepsilon$ and $\nabla \cdot \mathbf{b}^\varepsilon$ by the divergence-free conditions and boundary conditions. Thus it creates a loss of the $x$-derivative and a $y$-integration to the $y$-variable. Then the standard energy estimates do not work. To overcome this essential difficulty, inspired by recent results in [7, 14] that only need that the background tangential magnetic field $(1 + b) \geq \delta$, $\delta > 0$ has a lower positive bound instead of Oleinik’s monotonicity assumption on the tangential velocity.

We now apply the differential operator $\partial_y^2(|b| = 4)$ to the first two equations of (4.1), have

$$
(\partial_\tau - \partial_y^2 - \varepsilon \partial_y^2 + (u^\varepsilon + u^\varepsilon) \partial_y + v^\varepsilon \partial_y) \partial_x \mathbf{u}^\varepsilon + \partial_x \nabla \partial_x \mathbf{u}^\varepsilon
$$

$$
- (1 + b^\varepsilon) \partial_y \partial_x \mathbf{b}^\varepsilon - \partial_y \partial_x g^\varepsilon \partial_y \mathbf{b}^\varepsilon + \partial_y \partial_x v^\varepsilon \partial_y \mathbf{u}^\varepsilon = r_{u^\varepsilon}
$$

(5.35)

and

$$
(\partial_\tau - \partial_y^2 - \varepsilon \partial_y^2 + (u^\varepsilon + u^\varepsilon) \partial_y + v^\varepsilon \partial_y) \partial_x \mathbf{b}^\varepsilon + \partial_x \nabla \partial_x \mathbf{b}^\varepsilon
$$

$$
- (1 + b^\varepsilon) \partial_y \partial_x \mathbf{u}^\varepsilon - \partial_y \partial_x g^\varepsilon \partial_y \mathbf{u}^\varepsilon - \partial_y \partial_x v^\varepsilon \partial_y \mathbf{u}^\varepsilon = r_{b^\varepsilon},
$$

(5.36)

where

$$
\mathbf{r}_{u^\varepsilon} = - \sum_{0 < \beta < \beta} \partial_y \partial_x \mathbf{b}^\varepsilon \partial_x \mathbf{u}^\varepsilon - [\partial_x \mathbf{b}^\varepsilon, (u^\varepsilon + u^\varepsilon) \partial_y] \mathbf{u}^\varepsilon + [\partial_x \mathbf{u}^\varepsilon, (1 + b^\varepsilon) \partial_y] \mathbf{b}^\varepsilon
$$

$$
+ [\partial_y \partial_x \mathbf{b}^\varepsilon, \partial_y \mathbf{u}^\varepsilon - \partial_y \partial_x \mathbf{u}^\varepsilon]
$$

(5.37)

and

$$
\mathbf{r}_{b^\varepsilon} = - \sum_{0 < \beta < \beta} \partial_y \partial_x \mathbf{b}^\varepsilon \partial_x \mathbf{b}^\varepsilon - [\partial_x \mathbf{b}^\varepsilon, (u^\varepsilon + u^\varepsilon) \partial_y] \mathbf{b}^\varepsilon + [\partial_x \mathbf{b}^\varepsilon, (1 + b^\varepsilon) \partial_y] \mathbf{u}^\varepsilon
$$

$$
+ [\partial_y \partial_x \mathbf{b}^\varepsilon, \partial_y \mathbf{u}^\varepsilon - \partial_y \partial_x \mathbf{b}^\varepsilon]
$$

(5.38)

Exploiting the expression (5.37) and the commutator operator, the $L^2_{k+1}$-estimates of each terms in (5.37) can be controlled, then we can conclude the estimates of $\|\mathbf{r}_{u^\varepsilon}\|_{L^2_{k+1}}$ and $\|\mathbf{r}_{b^\varepsilon}\|_{L^2_{k+1}}$. We establish the estimate of the term $\|\mathbf{r}_{u^\varepsilon}\|_{L^2_{k+1}}$ by using the inequalities (2.2)-(2.3), we derive

$$
\|\mathbf{r}_{u^\varepsilon}\|_{L^2_{k+1}} \leq C(\|u^\varepsilon\|_{H^2_{k+1}} \|b^\varepsilon\|_{H^2_{k+1}} + \|(u^\varepsilon, b^\varepsilon)\|_{H^1_{k+1}}).
$$

(5.39)

The term $\|\mathbf{r}_{b^\varepsilon}\|_{L^2_{k+1}}$ can be estimated similarly,

$$
\|\mathbf{r}_{b^\varepsilon}\|_{L^2_{k+1}} \leq C(\|u^\varepsilon\|_{H^2_{k+1}} \|b^\varepsilon\|_{H^2_{k+1}} + \|(u^\varepsilon, b^\varepsilon)\|_{H^1_{k+1}}).
$$

(5.40)

Next, we consider the equations (5.35) and (5.36). It is obviously that the major difficulty derives from the terms

$$
\partial_y \partial_x \mathbf{u}^\varepsilon - \partial_y \partial_x g^\varepsilon \partial_y \mathbf{b}^\varepsilon
$$

$$
= -(\partial_y \partial_x + \partial_y \partial_x) (\partial_y^{-1} \partial_y \partial_x \mathbf{u}^\varepsilon) + \partial_y^{-1} \partial_y \partial_x g^\varepsilon \partial_y \mathbf{b}^\varepsilon
$$

(5.41)
where
\[
\partial_t^2 v^e \partial_t b^e - (\partial_t^2 g^e \partial_t u^e + \partial_t^4 g^e \partial_t u^e) = -(\partial_t^{-1} \partial_t^{\alpha+2} u^e) \partial_t b^e + (\partial_t u^e + \partial_t u^e)(\partial_t^{-1} \partial_t^{\alpha+2} b^e),
\] (5.42)
which imply the $5^{th}$-order tangential derivatives, and they cannot be controlled by the standard energy estimates.

To overcome this difficulty, inspired by recent results of [14], we depend on the following two main observations. One is that from the divergence-free condition $\partial_s b^e + \partial_s g^e = 0$, we give a stream function $\psi^e$ satisfies
\[
\partial_s \psi^e = b^e, \quad \partial_s \psi^e = -g^e, \quad \psi^e |_{y=0} = 0.
\] (5.43)
Then, using the equation (4.1), we can derive
\[
(\partial_t - \partial_t^2 - \varepsilon \partial_t^3 + (u^e + u^e) \partial_t + v^e \partial_s) \psi^e + v^e = 0.
\]
Applying differential operator $\partial_t^2$ to above equation as follows
\[
(\partial_t - \partial_t^2 - \varepsilon \partial_t^3 + (u^e + u^e) \partial_t + v^e \partial_s) \partial_t \psi^e + \partial_t v^e (1 + b^e) = r_{\psi^e},
\] (5.44)
where
\[
r_{\psi^e} = -[\partial_t^2, (u^e + u^e) \partial_s] \psi^e - \sum_{1 \leq |\beta| \leq 3} \partial_t^{3-\beta} v^e \partial_t^\beta \partial_s \psi^e.
\]

At the moment, by using the inequalities (2.2)-(2.3) and commutator operator, we can conclude the following estimate that $|\beta| < 4$,
\[
\|(y)^{k+1} r_{\psi^e} \|_{L^2} \leq \|(y)^{k+1}[\partial_t^2, (u^e + u^e) \partial_s] \psi^e \|_{L^2} + \|(y)^{k+1} \sum_{\beta < \beta} \partial_t^{3-\beta} v^e \partial_t^\beta \partial_s \psi^e \|_{L^2}
\]
\[
\leq \|(y)^{k+1} \sum_{\beta < \beta} \partial_t^{3-\beta} (u^e + u^e) \partial_t^\beta \partial_s \psi^e \|_{L^2} + \|(y)^{k+1} \sum_{\beta < \beta} \partial_t^{3-\beta} v^e \partial_t^\beta \partial_s \psi^e \|_{L^2}
\]
\[
\leq C(1 + ||u^e||_{H^3}) ||g^e||_{H^k_0} + ||u^e||_{H^3} ||b^e||_{H^k_0}
\]
\[
\leq C(1 + ||u^e||_{H^3}) ||b^e||_{H^k_0}.
\] (5.45)

We set $\xi_{u^e} = \frac{\partial_t u^e + \partial_t u^e}{1 + b^e}$ and $\xi_{b^e} = \frac{\partial_b b^e}{1 + b^e}$ and introduce the following two new unknown functions
\[
u^e := \partial_t^2 u^e - \frac{\partial_t u^e + \partial_t u^e}{1 + b^e} \partial_t \psi^e, \quad b^e := \partial_t^2 b^e - \frac{\partial_b b^e}{1 + b^e} \partial_t \psi^e.
\] (5.46)

On the other hand, we use the above two given unknown functions $(u^e, b^e)$ in (5.46) to deal with the loss regularity of $g^e = -\partial_t^{-1} \partial_s b^e$ by using the convection terms $-(1 + b^e) \partial_s b^e$ and $-(1 + b^e) \partial_s u^e$. More specifically,
\[
-(1 + b^e) \partial_s \partial_t b^e - \partial_t^2 g^e \partial_s b^e
\]
\[
= -(1 + b^e) \partial_s (b^e + \frac{\partial_t b^e}{1 + b^e} \partial_t \psi^e) - \partial_t^2 g^e \partial_s b^e
\]
\[AIMS Mathematics\]
where the initial date (5.35) and (5.36), we can derive the following equations of (5.35) and (5.36). We can also get the following initial and boundary conditions

\[
\begin{cases}
\quad \frac{\partial u^e}{\partial t} + (u^e + u^s)\partial_x + v^s\partial_y = r_{\psi^e, u^e}, \\
\quad \frac{\partial b^e}{\partial t} + (u^e + u^s)\partial_x + v^s\partial_y = r_{\psi^e, b^e},
\end{cases}
\]

(5.49)

where

\[
\begin{align*}
\quad r_{\psi^e, u^e} &= r_{\psi^e} - \xi_{\psi^e}r_{\psi} - \partial_\psi^e\psi^e(\partial_t - \partial_\psi^2 - \xi_{\psi^2})u^e, \\
\quad r_{\psi^e, b^e} &= r_{\psi^e} - \xi_{\psi^e}r_{\psi} - \partial_\psi^e\psi^e(\partial_t - \partial_\psi^2 - \xi_{\psi^2})b^e + (1 + b^e)\partial_x \xi_{\psi^e}\partial_\psi^2\psi^e,
\end{align*}
\]

(5.50)

We can also get the following initial and boundary conditions

\[
\begin{align*}
\quad u^e_{|t=0} &= \partial_\psi^2 u^e(0, x, y) - \frac{\partial_{u^e(0, x, y)} + \partial_{u^e(0, x, y)}}{1 + b^e} \int_0^x \partial_\psi^2 b^e(0, x, y)dy \bigg|_{y=0}, \\
\quad b^e_{|t=0} &= \partial_\psi^2 b^e(0, x, y) - \frac{\partial_{b^e(0, x, y)} + \partial_{b^e(0, x, y)}}{1 + b^e} \int_0^x \partial_\psi^2 b^e(0, x, y)dy \bigg|_{y=0}, \\
\quad u^e_{|y=0} &= 0, \quad b^e_{|y=0} = 0.
\end{align*}
\]

(5.51)

Finally, we derive the initial boundary value problem for \((u^e_{|t=0}, b^e_{|t=0})\) as follows

\[
\begin{align*}
\quad (\partial_t - \partial_\psi^2 - \xi_{\psi^2})u^e = r_{\psi^e, u^e}, \\
\quad (\partial_t - \partial_\psi^2 - \xi_{\psi^2})b^e = r_{\psi^e, b^e}, \\
\quad (u^e_{|t=0}, b^e_{|t=0}) = (u^e_{|t=0}, b^e_{|t=0}),
\end{align*}
\]

(5.52)

where the initial date \((u^e_{|t=0}, b^e_{|t=0})\) and \((r_{\psi^e, u^e}, r_{\psi^e, b^e})\) given by (5.50) and (5.51) respectively.

Moreover, since \(\psi = \partial_\psi^2 b^e\), \(\psi_{|y=0} = 0\), we deduce

\[
||\partial_\psi^2 \psi^e(t)||_{L^2(B; L^\infty(B))} \leq C||\partial_\psi^2 b^e(t)||_{L^2}.
\]

(5.53)

According to the expressions of \(\xi_{\psi^e}\) and \(\xi_{\psi^e}\), and the Sobolev embedding inequality, we derive that for \(1 \leq \lambda < k\),

\[
\begin{align*}
||\langle y \rangle^k \xi_{\psi^e}||_{L^\infty(B^1)} &= ||\langle y \rangle^k \frac{\partial_t u^e + \partial_x u^s}{1 + b^e}||_{L^\infty(B^1)} \\
&\leq ||\langle y \rangle^k \partial_t u^e||_{L^\infty(B^1)} + ||\langle y \rangle^k \partial_x u^s||_{L^\infty(B^1)} \\
&\leq C\delta^{-1} ||\langle y \rangle^k \partial_t u^e||_{H^0_{\delta}} + ||\langle y \rangle^{1-k}||_{L^\infty(B^1)}
\end{align*}
\]
Similarly, we can derive that
\[
\frac{\partial^2 \xi_{\nu'}}{\partial \nu' \partial \nu''} \leq C \delta^{-1}(\|u'\|_{H_{k+1}^3} + 1).
\] (5.54)

Analogously, we also have

\[
\begin{align*}
\|\psi^4 \xi_\nu \|_{L^2(B)} &\leq C \delta^{-1}\|\beta'\|_{H^3_{k+1}}, \\
\|\psi^4 \partial_\nu \xi_\nu \|_{L^2(B)} &\leq C \delta^{-1}\|u'\|_{H^4_{k+1}} + C \delta^{-2}(1 + \|u'\|_{H^4_{k+1}})\|\beta'\|_{H^4_{k+1}}, \\
\|\psi^4 \partial_\nu \xi_\nu \|_{L^2(B)} &\leq C \delta^{-1}(1 + \|u'\|_{H^4_{k+1}}) + C \delta^{-2}(1 + \|u'\|_{H^4_{k+1}})\|\beta'\|_{H^4_{k+1}}, \\
\|\partial^2 \xi_{\nu'} \|_{L^\infty(B)} &\leq C \delta^{-1}(1 + \|u'\|_{H^4_{k+1}}) + C \delta^{-2}(1 + \|u'\|_{H^4_{k+1}})\|\beta'\|_{H^4_{k+1}}, \\
\|\partial^2 \xi_{\nu'} \|_{L^\infty(B)} &\leq C \delta^{-1}(1 + \|u'\|_{H^4_{k+1}}) + C \delta^{-2}(1 + \|u'\|_{H^4_{k+1}})\|\beta'\|_{H^4_{k+1}}.
\end{align*}
\]
(5.55)

which combined with (5.39),(5.40), (5.45), (5.53) and (5.54), we can infer that for \(|\beta| = 4,
\]
\[
\|r_{\psi',\psi''}\|_{L^2_{k+1}} \leq C(\|u'\|_{H^4_{k+1}} + \|\beta'\|_{H^4_{k+1}})
\]
(5.56)

Similarly, we can derive that
\[
\|r_{\psi',\psi'}\|_{L^2_{k+1}} \leq C \delta^{-3}(1 + \|(u', b')\|_{H^5_{k+1}})\|(u', b')\|_{H^5_{k+1}}
\]
(5.57)
Lemma 5.4. We have the following estimate of \((u^e_\beta, b^e_\beta)\) given in (5.46),

\[
\sum_{\beta=4} \left( \frac{d}{dt} ||(u^e_\beta, b^e_\beta)(t)||^2_{L^2_{x,t}} + \epsilon ||\partial_x (u^e_\beta, b^e_\beta)(t)||^2_{L^2_{x,t}} + ||\partial_t (u^e_\beta, b^e_\beta)(t)||^2_{L^2_{x,t}} \right)
\leq C\delta^{-6} (1 + ||(u^e, b^e)||^2_{H^1_{t,x}})^2 + C\delta^{-1} (1 + ||(u^e, b^e)||_{H^1_{t,x}})^4 \sum_{\beta=4} ||(u^e_\beta, b^e_\beta)||^2_{L^2_{x,t}}
\]

\[
+ \frac{\epsilon \delta}{4} \sum_{\beta=4} ||\partial_x \partial_x^2 b^e||^2_{L^2_{t,x}}.
\]

(5.58)

Proof. Similar to the proof of Lemma 5.3, multiplying (5.49)_{1,2} by \(\langle y \rangle^{2(k+1)} u^e_\beta\) and \(\langle y \rangle^{2(k+1)} b^e_\beta\) respectively, and integrating it by parts over \(Q_T\), we deduce that

\[
\sum_{\beta=4} \left( \frac{1}{2} \frac{d}{dt} ||(u^e_\beta, b^e_\beta)(t)||^2_{L^2_{x,t}} + \epsilon ||\partial_x (u^e_\beta, b^e_\beta)(t)||^2_{L^2_{x,t}} + ||\partial_t (u^e_\beta, b^e_\beta)(t)||^2_{L^2_{x,t}} \right)
\]

\[
= \sum_{\beta=4} \left( 2(k+1) \int_{\mathbb{R}^2} \langle y \rangle^{2(k+1)-1} \psi \langle u^e_\beta \rangle^2 + ||b^e_\beta \rangle^2 \right) dx dy - \int_{\mathbb{R}^2} \langle y \rangle^{2(k+1)} b^e(u^e_\beta, b^e_\beta) dx dy
\]

\[
+ 2(k+1) \int_{\mathbb{R}^2} \langle y \rangle^{2(k+1)-1} (\partial_x u^e_\beta \partial_x b^e_\beta + \partial_t b^e_\beta \partial_t b^e_\beta) dx dy
\]

\[
+ \int_{\mathbb{R}^2} \langle y \rangle^{2(k+1)} (r_{\psi, u^e_\beta} u^e_\beta + r_{\psi, b^e_\beta} b^e_\beta) dx dy = \sum_{i=1}^4 \tilde{J}_i,
\]

(5.59)

where we have used the boundary conditions (5.52) and \((v^e, b^e)|_{\gamma=0} = 0\).

Next, we deal with the estimates of the right-hand side terms of (5.59) as follows. By the Sobolev inequality, we first establish the estimate of term \(\tilde{J}_1\),

\[
|\tilde{J}_1| \leq \epsilon ||(u^e_\beta, b^e_\beta)||^2_{L^2_{t,x}} \leq C ||u^e||_{H^0} ||(u^e_\beta, b^e_\beta)||^2_{L^2_{t,x}},
\]

(5.60)

Similar to (5.60), exploiting the Sobolev inequality and Young inequality, we can obtain

\[
|\tilde{J}_2| \leq C ||b^e||_{H^0} ||(u^e_\beta, b^e_\beta)||^2_{L^2_{t,x}},
\]

(5.61)

and

\[
|\tilde{J}_3| \leq \frac{1}{2} ||\partial_x (u^e_\beta, b^e_\beta)||^2_{L^2_{t,x}} + C ||(u^e_\beta, b^e_\beta)||^2_{L^2_{t,x}}.
\]

(5.62)

For \(\tilde{J}_4\), using (5.66)-(5.57) and Young inequality, we infer

\[
|\tilde{J}_4| \leq ||r_{\psi, u^e_\beta}||_{L^2_{t,x}} ||u^e_\beta||^2_{L^2_{t,x}} + ||r_{\psi, b^e_\beta}||_{L^2_{t,x}} ||b^e_\beta||^2_{L^2_{t,x}}
\]

\[
\leq C \left( \delta^{-3} (1 + ||u^e, b^e||_{H^1_{t,x}})^2 ||(u^e, b^e)||_{H^1_{t,x}} \right) + C \epsilon ||\partial_x b^e_\beta||_{L^2_{t,x}} ||(u^e, b^e)||_{H^1_{t,x}} ||(u^e_\beta, b^e_\beta)||_{L^2_{t,x}}
\]

\[
\leq C \delta^{-6} (1 + ||u^e, b^e||_{H^1_{t,x}})^2 + \delta^{-1} (1 + ||u^e, b^e||_{H^1_{t,x}})^4 ||(u^e_\beta, b^e_\beta)||^2_{L^2_{t,x}}
\]

(5.57)
Substituting (5.60)-(5.63) into (5.59) and taking the summation over all \( |\beta| = 4 \) in (5.59), we can derive the desired result (5.58). The proof is therefore completed.

Up to now, we have completed the main estimates of the solutions \((u^\varepsilon, b^\varepsilon)\) for (4.1). However, Lemma 5.4 gives the estimates of \((u_\beta^\varepsilon, b_\beta^\varepsilon)\), thus we need to show the equivalence in the \(L^2_{k+1}\)-norm between \((\partial_t^\varepsilon u^\varepsilon, \partial_t^\varepsilon b^\varepsilon)\) and \((u_\beta^\varepsilon, b_\beta^\varepsilon)\) given by (5.46).

**Lemma 5.5.** If the smooth function \((u^\varepsilon, b^\varepsilon)\) satisfies the problem (4.1) in \([0, T]\) and tangential magnetic field has a lower positive bound, then for \( \forall t \in [0, T] \), \( k \geq 1, l \geq 0 \), and the equality \((u_\beta^\varepsilon, b_\beta^\varepsilon)\) with \(|\beta| = 4\) defined by (5.46), we conclude

\[
\gamma(t)^{-1} \|(\partial_t^\varepsilon u^\varepsilon, \partial_t^\varepsilon b^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} \leq \|(u_\beta^\varepsilon, b_\beta^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} \leq \gamma(t) \|(\partial_t^\varepsilon u^\varepsilon, \partial_t^\varepsilon b^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)},
\]

(5.64)

and

\[
\|(\partial_t^\varepsilon u^\varepsilon, \partial_t^\varepsilon b^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} \leq \|(\partial_t^\varepsilon u^\varepsilon, \partial_t^\varepsilon b^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} + \|(\partial_t^\varepsilon b^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} + \|(\partial_t b^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} + \|(\partial_t^\varepsilon b^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} + \|(\partial_t^\varepsilon b^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} + \|(\partial_t b^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} + \|(\partial_t^\varepsilon b^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)},
\]

(5.65)

where

\[
\gamma(t) = \delta^{-1}(1 + \|(\partial_t u^\varepsilon, b^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)}).
\]

(5.66)

**Proof.** According to the definitions of \(u_\beta^\varepsilon\) and \(b_\beta^\varepsilon\) in (5.46), and using the equalities (2.2)-(2.3), we can derive from (5.53)

\[
\|(u_\beta^\varepsilon, b_\beta^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} \leq \|(\partial_t^\varepsilon u^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} + \|(\partial_t b^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} + \|(\partial_t^\varepsilon b^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} + \|(\psi^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} + \|(\partial_t^\varepsilon \psi^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} + \|(\partial_t^\varepsilon \psi^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} + \|(\partial_t^\varepsilon \psi^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)}
\]

(5.67)

and,

\[
\|(b_\beta^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} \leq \|(\partial_t^\varepsilon b^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} + \|(\partial_t b^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} + \|(\partial_t^\varepsilon b^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} + \|(\partial_t^\varepsilon b^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} + \|(\partial_t b^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} + \|(\partial_t^\varepsilon b^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} + \|(\partial_t^\varepsilon b^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} + \|(\partial_t b^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} + \|(\partial_t^\varepsilon b^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)}
\]

(5.68)

Therefore, we derive

\[
\|(u_\beta^\varepsilon, b_\beta^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)} \leq \gamma(t)^{-1} \|(\partial_t^\varepsilon u^\varepsilon, \partial_t^\varepsilon b^\varepsilon)\|_{L^2_{k+1}(\mathbb{R}^2)}
\]

(5.69)

On the other hand, since the equality \(\partial_t^\varepsilon \psi^\varepsilon = b^\varepsilon\) and expression of \(b_\beta^\varepsilon\) in (5.46),

\[
b_\beta^\varepsilon := \partial_t^\varepsilon b^\varepsilon - \frac{\partial_t b^\varepsilon}{1 + b^\varepsilon} \partial_t^\varepsilon \psi^\varepsilon = (1 + b^\varepsilon) \partial_t^\varepsilon \psi^\varepsilon,
\]

which gives that by \(\partial_t^\varepsilon \psi^\varepsilon|_{y=0} = 0\),

\[
\partial_t^\varepsilon \psi^\varepsilon(t, x, y) = (1 + b^\varepsilon(t, x, y)) \int_0^y \frac{b_\beta^\varepsilon(t, x, \tilde{y})}{1 + b^\varepsilon(t, x, \tilde{y})} d\tilde{y},
\]

(5.70)
which combined with (5.46), we attain

\[
\begin{align*}
\partial_t^k u^\varepsilon &= u_{\beta}^\varepsilon + (\partial_x u^\varepsilon + \partial_x u^\varepsilon) \cdot \int_0^\gamma \frac{b_\beta^\varepsilon(t,x,y)}{1 + b_\beta^\varepsilon(t,x,y)} dy, \\
\partial_t^k b^\varepsilon &= b_{\beta}^\varepsilon + \partial_x b^\varepsilon \cdot \int_0^\gamma \frac{b_\beta^\varepsilon(t,x,y)}{1 + b_\beta^\varepsilon(t,x,y)} dy.
\end{align*}
\]

(5.71)

Then, we have for \( k \geq 1 \),

\[
\|\partial_t^k u^\varepsilon\|_{L^2_{x,t}} \leq \|u_{\beta}^\varepsilon\|_{L^2_{x,t}} + \|\partial_x u^\varepsilon\|_{L^2_{x,t}} \|y\|^{-1} \int_0^\gamma \frac{b_\beta^\varepsilon(t,x,y)}{1 + b_\beta^\varepsilon(t,x,y)} dy \|u\|_{L^2}
\]

\[
+ \|\gamma y\| \int_0^\gamma \frac{b_\beta^\varepsilon(t,x,y)}{1 + b_\beta^\varepsilon(t,x,y)} dy \|u\|_{L^2} 
\]

\[
\leq \|u_{\beta}^\varepsilon\|_{L^2_{x,t}} + C\delta^{-1}(1 + \|\partial_x u^\varepsilon\|_{L^2_{x,t}})\|b_{\beta}^\varepsilon\|_{L^2}. 
\]

(5.72)

Also,

\[
\|\partial_t^k b^\varepsilon\|_{L^2_{x,t}} \leq \|b_{\beta}^\varepsilon\|_{L^2_{x,t}} + C\delta^{-1}||\partial_x b^\varepsilon||_{L^2_{x,t}} \|b_{\beta}^\varepsilon\|_{L^2_{x,t}}, 
\]

(5.73)

which gives

\[
\|\left(\partial_t^k u^\varepsilon, \partial_t^k b^\varepsilon\right)\|_{L^2_{x,t}} \leq \gamma(t)^{-1} \|\left(u_{\beta}^\varepsilon, b_{\beta}^\varepsilon\right)\|_{L^2_{x,t}},
\]

(5.74)

provided that \( \gamma(t) \) given (5.66). Hence, combining (5.69) with (5.74) implies (5.64).

Similar to (5.72), we also get the desired result (5.65). The proof is thus completed.

In this part, we will complete the proof of theorem 2.1. Similar to the proof of Proposition 3.6 in [14], (2.1), we have

\[
\|y\|^k \partial_t^k \partial_x(y u^\varepsilon, b^\varepsilon)\|_{L^2} \leq \delta^{-1}, \text{ for } i = 1, 2, t \in [0, T],
\]

(5.75)

which combined with (5.46) and (5.66), it follows that for \( \delta \in (0, 1) \) small enough,

\[
\gamma(t) = \delta^{-1}(1 + \|\partial_x u^\varepsilon\|_{L^2_{x,t}}) \leq 2\delta^{-2};
\]

(5.76)

Then, since the operator \( D^x = \partial_t^k \partial_x \), from (5.64), we discover

\[
\|\left(u^\varepsilon, b^\varepsilon\right)\|_{H^3_{x,t}}^2 = \sum_{|\alpha| \leq 4, |\beta| \leq 3} \|D^x(u^\varepsilon, b^\varepsilon)(t)\|_{L^2_{x,t}}^2 + \sum_{|\beta| = 4} \|\partial_t^k(u^\varepsilon, b^\varepsilon)(t)\|_{L^2_{x,t}}^2
\]

\[
\leq \sum_{|\alpha| \leq 4, |\beta| \leq 3} \|D^x(u^\varepsilon, b^\varepsilon)(t)\|_{L^2_{x,t}}^2 + 4\delta^{-2} \sum_{|\beta| = 4} \|\partial_t^k(u^\varepsilon, b^\varepsilon)(t)\|_{L^2_{x,t}}^2.
\]

(5.77)

In this position, we show the energy estimate of the approximate solutions \( (u^\varepsilon, b^\varepsilon) \). Collecting some established estimates (5.1), (5.58), (5.65) and (5.76) and adding together, then inserting (5.77) into the resultant and using Young inequality, which implies

\[
\frac{d}{dt} \left( \sum_{|\alpha| \leq 4, |\beta| \leq 3} \|D^x(u^\varepsilon, b^\varepsilon)(t)\|_{L^2_{x,t}}^2 + 4\delta^{-2} \sum_{|\beta| = 4} \|\partial_t^k(u^\varepsilon, b^\varepsilon)(t)\|_{L^2_{x,t}}^2 \right)
\]
Under the assumptions of Theorem 2.1, there exists a positive constant $C$, may be depend

$\leq C\delta^{-6}(1 + \|u^\varepsilon, b^\varepsilon\|_{H_{\varepsilon,l}^k})^2 + C\delta^{-2}(1 + \|u^\varepsilon, b^\varepsilon\|_{H_{\varepsilon,l}^k})^2 \sum_{[\beta]=4} \|u_{\beta}^\varepsilon, b_{\beta}^\varepsilon\|_{l_2}^2$

$\leq C\delta^{-6}\left( \sum_{[\alpha]=4, [\beta]=3} \|D^{\alpha}(u^\varepsilon, b^\varepsilon)(t)\|_{l_2}^2 + 4\delta^{-2}\sum_{[\beta]=4} \|u_{\beta}^\varepsilon, b_{\beta}^\varepsilon\|_{l_2}^2 \right)^3$

$+ C(\delta^{-2} + \delta^{-6}).$ \hspace{1cm} (5.78)

Define

$F_0 := \sum_{[\alpha]=4, [\beta]=3} \|D^{\alpha}(u^\varepsilon, b^\varepsilon)(0)\|_{l_2}^2 + 4\delta^{-2}\sum_{[\beta]=4} \|u_{\beta}^\varepsilon, b_{\beta}^\varepsilon\|_{l_2}^2.$ \hspace{1cm} (5.79)

Consequently, applying the nonlinear Gronwall inequality (Theorem 2, P362, [18]) in (5.78), we have

$\sum_{[\alpha]=4, [\beta]=3} \|D^{\alpha}(u^\varepsilon, b^\varepsilon)(t)\|_{l_2}^2 + 4\delta^{-2}\sum_{[\beta]=4} \|u_{\beta}^\varepsilon, b_{\beta}^\varepsilon\|_{l_2}^2$

$\leq (F_0 + (\delta^{-2} + \delta^{-6})t)[1 - 2C\delta^{-6}(F_0 + (\delta^{-2} + \delta^{-6})t)^2]^{-\frac{1}{2}}.$ \hspace{1cm} (5.80)

Up to now, we have the following lemma.

**Lemma 5.6.** Under the assumptions of Theorem 2.1, there exists a positive constant $C$, may be depend

$\|(u^\varepsilon, b^\varepsilon)(t)\|_{H_{\varepsilon,l}^k}^2 \leq 16\delta^{-2}\|(u, b)(0)\|_{H_{\varepsilon,l}^k}^2, \quad \forall t \in [0, T].$ \hspace{1cm} (5.81)

**Proof.** First, invoking (5.77) and (5.80), we can derive that

$\|(u^\varepsilon, b^\varepsilon)(t)\|_{H_{\varepsilon,l}^k}^2 \leq (F_0 + (\delta^{-2} + \delta^{-6})t)[1 - 2C\delta^{-6}(F_0 + (\delta^{-2} + \delta^{-6})t)^2]^{-\frac{1}{2}}.$ \hspace{1cm} (5.82)

Then, combining (5.79), (5.64), (5.66) with (4.22), we can lead to

$F_0 \leq C\left( \sum_{[\alpha]=4, [\beta]=3} \|D^{\alpha}(u, b)(0)\|_{l_2}^2 + 4\delta^{-2}\sum_{[\beta]=4} \|(u_{\beta}, b_{\beta})(0)\|_{l_2}^2 \right)$

$\leq C\delta^{-2}\|(u, b)(0)\|_{H_{\varepsilon,l}^k}^2.$ \hspace{1cm} (5.83)

Hence, we can conclude the uniform estimates with respect to $\varepsilon \in (0, 1)$ and $\forall t \in [0, T]$

$\|(u^\varepsilon, b^\varepsilon)(t)\|_{H_{\varepsilon,l}^k}^2 \leq 16\delta^{-2}\|(u, b)(0)\|_{H_{\varepsilon,l}^k}^2,$

provide that $T$ be determined by (5.82) and (5.83) such that

$T = \min\left\{ \frac{3C\|(u, b)(0)\|_{H_{\varepsilon,l}^k}^2}{(1 + \delta^{-4})}, \frac{\delta^{10}}{64C\|(u, b)(0)\|_{H_{\varepsilon,l}^k}^4} \right\}.$ \hspace{1cm} (5.84)
The proof is thus completed. □

Convergence and Consistency

Using evolution equation (4.1) and uniform $H^4_{k+l}$ bound in (5.81), we conclude that $(\partial_t u^\varepsilon, \partial_t b^\varepsilon)$ is uniformly (in $\varepsilon$) bounded in $L^\infty([0, T]; H_{k+l}^4)$. By the Lions-Aubin Lemma and the compact embedding of $H_{k+l}^4$ in $H_{k+l}^{1-\tilde{\delta}}$, for $0 < \tilde{\delta} < 1$. Then taking a subsequence, as $\varepsilon_k \to 0^+$,

$$(u^\varepsilon, b^\varepsilon) \rightharpoonup (u, b) \quad \text{in} \quad L^\infty([0, T]; H_{k+l}^4) \quad \text{and} \quad (u^\varepsilon, b^\varepsilon) \to (u, b) \quad \text{in} \quad C([0, T]; H_{k+l+1}^{1-\tilde{\delta}}).$$

Applying the local uniform convergence of $(\partial^4_{x} u^\varepsilon, \partial^4_{x} b^\varepsilon)$, we have the following pointwise convergence of $(v^\varepsilon, g^\varepsilon)$: as $\varepsilon_k \to 0^+$,

$$(v^\varepsilon, g^\varepsilon) = \left( - \int_0^y \partial_x u^\varepsilon \, dy, - \int_0^y \partial_x b^\varepsilon \, dy \right) \to \left( - \int_0^y \partial_x u \, dy, - \int_0^y \partial_x b \, dy \right) =: (v, g). \quad (5.85)$$

Now, we pass the limit in the problem (4.1) and conclude that $(u, v, b, g)$ solve the original problem (3.4). Hence, we finish the proof of Theorem 2.1.

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Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares to have no competing interests.

References


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