Dynamic analysis of a stochastic vector-borne model with direct transmission and media coverage

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Abstract: This paper presents a stochastic vector-borne epidemic model with direct transmission and media coverage. It proves the existence and uniqueness of positive solutions by constructing a suitable Lyapunov function. Immediately after that, we study the transmission mechanism of vector-borne diseases and give threshold conditions for disease extinction and persistence, in addition to which we show that the model has a stationary distribution determined by the synergistic action of two parameters, i.e., the existence of a stationary distribution is unique under specific condition. Finally, a stochastic model describing the dynamics of vector-borne diseases is numerically simulated to illustrate our mathematical findings.

Keywords: Vector-borne disease; Direct transmission; Media coverage; Stationary distribution
Mathematics Subject Classification: 92B05, 60G51, 60G57

1. Introduction

Vector-borne disease seriously threatens world health, usually caused by vector-borne parasites, viruses, and bacteria transmitting pathogens between humans or from animals to humans. According to the World Health Organization, the disease accounts for 17% of all infectious and has caused 700,000 deaths annually [1]. Despite scientific and technological advances and growing affluence in all regions, vector-borne diseases remain one of the leading causes of global disease. Mathematical modeling has become an essential method for studying epidemic. Since the first modeling of malaria transmission by Ross [2] and subsequent modifications by MacDonald [3], a series of vector-borne disease models have been proposed [4–7]. Various disease models based on influencing factors (e.g., time delay, vaccination, age structure, etc.) have been extensively studied [8–11].

It’s commonly known that direct and indirect transmissions are two significant ways of the
spread of various diseases. Although indirect transmission is not negligible, vector-borne diseases are often transmitted directly through blood transfusions, organ transplantation, laboratory exposure, or mother-to-baby during pregnancy, childbirth, and breastfeeding. It is worth noting that zika can be transmitted through sexual contact[12]. Thus, direct transmission plays a vital role in the dynamics of vector-borne diseases and has attracted widespread attention [13–16]. In the deterministic model proposed by Wei et al.[16], the host population is assumed to be divided into three subpopulations, i.e., susceptible, vector-borne infected, and recovered individuals. The infected individuals will not relapse once recovered, i.e., the recovered individuals will not become susceptible or infected. Let \( S(t) \), \( I(t) \), and \( R(t) \) be the numbers of susceptible, infected, and recovered individuals at time \( t \). The vector population is divided into two parts, i.e., susceptible and infected vectors, denoted by \( M(t) \) and \( V(t) \) as the corresponding numbers at time \( t \). Once infected without recovering, the vectors will carry the virus for life. The newly recruited vectors are susceptible when vertical transmission is ignored. On the other hand, media coverage is a crucial factor in controlling the spread of epidemics [17]. Through the media, it is helpful to understand the progress of the epidemic and provide beneficial guidance [18]. Many scholars have studied the impact of media coverage on disease transmission through mathematical modeling [19, 20].

Based on the above discussion, we introduce media coverage into the epidemic model and investigate the dynamic of Vector-borne diseases with direct transmission. Let \( \beta_1 \) be the transmission rate without media intervention, and \( \beta_2/(m + I) \) be the effect of media coverage on transmission, where \( \beta_1 > \beta_2 \), and \( m \) measures how quickly people react to media reports [21]. During the spread of the vector-borne epidemic, the susceptible can be infected through two transmission rates: the rate denoted by \( \beta_3 \) from an infected vector to a susceptible person, and the one denoted by \( \beta_4 \) from an infected person to a susceptible vector. Then, we propose a vector-borne model with direct transmission and media coverage as follows

\[
\begin{align*}
\frac{dS}{dt} &= \left( \Lambda_1 - \left( \beta_1 - \frac{\beta_2 I}{m + I} \right) \frac{SI}{1 + \alpha_1 I} - \frac{\beta_3 SV}{1 + \alpha_2 V} - \frac{d_1 S}{I} \right) dt, \\
\frac{dI}{dt} &= \left( \beta_1 - \frac{\beta_2 I}{m + I} \right) \frac{SI}{1 + \alpha_1 I} + \frac{\beta_3 SV}{1 + \alpha_2 V} - (\mu + \gamma) I dt, \\
\frac{dR}{dt} &= (\gamma I - d_1 R) dt, \\
\frac{dM}{dt} &= \frac{\Lambda_2 - \beta_4 MI}{1 + \alpha_3 I} - d_2 M \right) dt, \\
\frac{dV}{dt} &= \left( \frac{\beta_4 MI}{1 + \alpha_3 I} - d_2 V \right) dt,
\end{align*}
\]

where \( \Lambda_i \), \( d_i (i = 1, 2) \), and \( \mu \) are the recruitment, natural, and disease-related death rates of people and vector population, \( \alpha_i (i = 1, 2, 3) \) are the saturated constants during different transmission processes, and \( \gamma \) is the recovery rate of infected people. Here, \( \beta_1 SI \), \( \beta_3 SV \), and \( \beta_4 MI \) measure the contagiousness of the vector-borne disease, and \( 1/(1 + \alpha_1 I) \), \( 1/(1 + \alpha_2 V) \), \( 1/(1 + \alpha_3 I) \) reflect the behavioral change of susceptible individuals. The basic reproduction number is \( R_0 = \frac{\beta_1 \Lambda_1}{d_1(d_1 + \gamma + \mu)} + \frac{\beta_2 \beta_3 \Lambda_1 \Lambda_2}{d_1 d_2 (d_1 + \gamma + \mu)} \), which determines the epidemic occurs or not. If \( R_0 < 1 \), system (1.1) has a unique disease-free equilibrium \( E_0 = \left( \frac{\Lambda_1}{d_1}, 0, 0, \frac{\Lambda_2}{d_2}, 0 \right) \). This represents no infected individuals in either
population. If $R_0 > 1$, then model (1.1) has two equilibria: a disease-free equilibrium $E_0$ and an endemic equilibrium $E^* = (S^*, I^*, R^*, M^*, V^*)$. This means that some individuals of both populations have been infected.

In the real world, random fluctuations are essential to ecosystems [22–24]. Random factors, such as temperature and humidity, inevitably affect the epidemic's spread. Many stochastic models have been studied in recent years [25–27]. Considering the complex environmental changes, Liu and Jiang claimed that the random perturbation may depend on the square of the state variables $S$ and $I$ in the system [28, 29]. Recently, nonlinear perturbations have received much attention [30–32]. In addition to this, Sometimes ecosystems are also affected by violent random perturbations such as typhoons and tsunamis. To reflect reality better, Levy jumps were introduced in the model [33, 34]. However, this noise differs in detail and often leads to different results. It is worth noting that in the model of vector-borne diseases, Jovanović and Krstić [35] proposed that the random perturbation is proportional to the distance. Ran et al. [36] studied the dynamics of a stochastic vector-borne model with age structure and saturation incidence, considering the environmental noise on mosquito bite rate and transmission rate between vector and host. Son et al. [37] provided another stochastic vector-borne model with direct transmission, in which environmental noise affects the mortality of hosts and vectors. We don’t want to add complex perturbations to make the model unmanageable; simple perturbations are more likely to reveal the inherent nature of the model. In our work, suppose that the environmental white noise is proportional to the number of subpopulations [38, 39]. Next, we extend the deterministic model (1.1) to a stochastic model. The recovered class is decoupled from the others in the model and then neglected. Then, we propose the following stochastic model

\[
\begin{align*}
\text{d}S &= \left( \Lambda_1 - \left( \frac{\beta_1}{m+I} \right) \frac{SI}{1+\alpha_1I} - \frac{\beta_3SV}{1+\alpha_2V} - d_1S \right) dt + \sigma_1 S dB_1(t), \\
\text{d}I &= \left( \left( \frac{\beta_2}{m+I} \right) \frac{SI}{1+\alpha_1I} + \frac{\beta_3SV}{1+\alpha_2V} - (\mu + d_1 + \gamma)I \right) dt + \sigma_2 I dB_2(t), \\
\text{d}M &= \left( \Lambda_2 - \frac{\beta_4MI}{1+\alpha_3I} - d_2M \right) dt + \sigma_3 M dB_3(t), \\
\text{d}V &= \left( \frac{\beta_4MI}{1+\alpha_3I} - d_2V \right) dt + \sigma_4 V dB_4(t),
\end{align*}
\]  

\hspace{1cm} (1.2)

where $B_i(t)$ ($i = 1, 2, 3, 4$) are independent standard Brownian motions, $\sigma_i$ ($i = 1, 2, 3, 4$) represent the white noise intensity, and the remaining parameters are the same as model (1.1).

The rest of this paper is organized as follows. Section 2 reviews some basic concepts and valuable lemmas used later. The uniqueness and positivity of the solution are proved in Section 3. Section 4 provides sufficient conditions for determining whether a disease is extinct. In Section 5, we explore the persistence in mean. In Section 6, we prove the existence of a unique ergodic stationary distribution under certain conditions. In Section 7, we validate the analysis results through numerical simulations. A brief conclusion is given in the last section.
2. Preliminaries

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfying the usual conditions (i.e., it is increasing and right continuous while \(\mathcal{F}_0\) contains all \(\mathbb{P}\)-null sets). Denote \(\mathbb{R}^n_+ = \{y \in \mathbb{R}^n : y_i > 0, 1 \leq i \leq n\}\). Consider an \(n\)-dimensional stochastic differential equation of the following form [40]

\[
dy(t) = f(y(t), t)dt + g(y(t), t)dB(t)
\]

with initial value \(y(0) = y_0 \in \mathbb{R}^n\), where \(B(t)\) denotes an \(n\)-dimensional standard Brownian motion defined on the complete probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\). Define a differential operator \(\mathcal{L}\) of Eq. (2.1) as follows

\[
\mathcal{L} = \frac{\partial}{\partial t} + \sum_{i=1}^n f_i(y, t) \frac{\partial}{\partial y_i} + \frac{1}{2} \sum_{i,j=1}^n \left[ g^T(y, t)g(y, t) \right]_{ij} \frac{\partial^2}{\partial y_i \partial y_j}.
\]

If \(\mathcal{L}\) acts on a nonnegative function function \(\mathcal{F} \in C^{2,1}(\mathbb{R}^n \times [0, \infty); \mathbb{R}_+)\), then

\[
\mathcal{L}\mathcal{F}(y, t) = \mathcal{F}_t(y, t) + \mathcal{F}_y(y, t)f(y, t) + \frac{1}{2} \text{trace}\left\{ g^T(y, t)\mathcal{F}_{yy}(y, t)g(y, t) \right\}.
\]

where \(\mathcal{F}_t = \frac{\partial \mathcal{F}}{\partial t}, \mathcal{F}_y = (\frac{\partial \mathcal{F}}{\partial y_1}, \ldots, \frac{\partial \mathcal{F}}{\partial y_n}), \mathcal{F}_{yy} = (\frac{\partial^2 \mathcal{F}}{\partial y_i \partial y_j})_{n \times n}\). By Itô’s formula, it follows that

\[
d\mathcal{F}(y(t), t) = \mathcal{L}\mathcal{F}(y(t), t)dt + \mathcal{F}_y(y(t), t)g(y(t), t)dB(t), \quad y(t) \in \mathbb{R}^n.
\]

Lemma 1. (Strong Law of Large Numbers, [41]) Let \(M = \{M_t\}_{t \geq 0}\) be a real-value continuous local martingale vanishing at \(t = 0\). Then

\[
\lim_{t \to \infty} \langle M, M \rangle_t = \infty \quad \text{a.s.} \quad \Rightarrow \lim_{t \to \infty} \frac{M_t}{\langle M, M \rangle_t} = 0. \quad \text{a.s.}
\]

\[
\limsup_{t \to \infty} \frac{\langle M, M \rangle_t}{t} < \infty \quad \text{a.s.} \quad \Rightarrow \lim_{t \to \infty} \frac{M_t}{t} = 0. \quad \text{a.s.}
\]

Let \(Y(t)\) be a regular time-homogeneous Markov process in \(\mathbb{R}^n\) in the following form

\[
dY(t) = a(Y)dt + \sum_{i=1}^k \sigma_i dB_i(t),
\]

where the diffusion matrix \(\tilde{A}(Y) = (b_{ij}(y))\) and \(b_{ij}(y) = \sum_{r=1}^k \sigma^r_i(y)\sigma^r_j(y)\).

Lemma 2. ([42]) The Markov process \(Y(t)\) has a unique stationary distribution \(\pi(\cdot)\) if there is a bounded domain \(D \in \mathbb{R}^n\) with a regular boundary such that its closure \(\bar{D} \in \mathbb{R}^n\) has the following properties

(i) In the open domain \(D\) and some of its neighbors, the smallest eigenvalue of the diffusion matrix \(A(t)\) is set far from zero.

(ii) If \(y \in \mathbb{R}^n \setminus D\), the mean time \(\tau\) at which a path issuing from \(y\) reaches the set \(D\) is finite, and \(\sup_{y \in \mathbb{R}^n} E\tau^t < \infty\) for every compact subset. Moreover, if \(f(\cdot)\) is a function integrable concerning the measure \(\pi\), then

\[
P\left\{ \limsup_{T \to \infty} \frac{1}{T} \int_0^T f(Y(t))dt = \int_{\mathbb{R}^n} f(y)\pi(dy) \right\} = 1.
\]
3. Uniqueness and positive of solution

Theorem 1. For a given initial value \( Y(0) = (S(0), I(0), M(0), V(0)) \in \mathbb{R}^4_+ \), the solution \( Y(t) = (S(t), I(t), M(t), V(t)) \) of model (1.2) is unique on \( t \geq 0 \) and will maintain in \( \mathbb{R}^4_+ \) with probability one.

Proof. For a given initial value \( (S(0), I(0), M(0), V(0)) \in \mathbb{R}^4_+ \), the coefficient in the model (1.2) satisfies the local Lipschitz continuity condition. Hence, there is a unique local solution when \( t \in [0, \tau_e) \), where \( \tau_e \) is the explosion time [43, 44]. To obtain the global property of the solution, we need to prove \( \tau_e = \infty \) a.s. Suppose that \( \tau_0 \geq 1 \) is enough large such that \( S(0), I(0), M(0) \) and \( V(0) \) all lie within the interval \([1/k_0, k_0]\). For each integer \( k \geq k_0 \), define a stopping time

\[
\tau_k = \inf\{t \in [0, \tau_e) : \min\{S(t), I(t), M(t), V(t)\} \leq 1/k \text{ or } \max\{S(t), I(t), M(t), V(t)\} \geq k\}, \quad (3.1)
\]

where \( \emptyset \) is an empty set and \( \inf \emptyset = \infty \). It can be seen that \( \tau_k \) increases as \( k \to \infty \), and \( \tau_\infty = \lim_{k \to \infty} \tau_k \) with \( 0 \leq \tau_\infty \leq \tau_e \) a.s. In other words, if \( \tau_e = \infty \) a.s. does not hold, there must have constants \( T, k_1 > 0 \) and \( \epsilon \in (0, 1) \) such that \( P\{\tau_k \leq T\} > \epsilon \) for all \( k \geq k_1 \). Define a \( C^2 \)-function \( W : \mathbb{R}^4 \to \mathbb{R}_+ \) and

\[
W(S(t), I(t), M(t), V(t)) = (S(t) - 1 - \log S(t)) + (I(t) - 1 - \log I(t)) + (M(t) - 1 - \log M(t))
+ (V(t) - 1 - \log V(t)). \quad (3.2)
\]

Obviously, \( W \) is a non-negative function. Applying Itô’s formula to (3.2) yields

\[
dW(S(t), I(t), M(t), V(t)) = \left[\left(1 - \frac{1}{S}\right) \left(\Lambda_1 - (\beta_1 - \frac{\beta_2 I}{m + I})\frac{SI}{1 + \alpha_1 I} - \frac{\beta_3 SV}{1 + \alpha_2 V} - d_1 S\right) + \frac{1}{2} \sigma^2_1 \right] dt
+ \left(1 - \frac{1}{I}\right) \left(\beta_1 - \frac{\beta_2 I}{m + I}\right) \left(\frac{SI}{1 + \alpha_1 I} + \frac{\beta_3 SV}{1 + \alpha_2 V} - (\mu + d_1 + \gamma) I\right) + \frac{1}{2} \sigma^2_2 \right] dB_1(t)
+ \frac{1}{V} \left(\Lambda_2 - \frac{\beta_4 MI}{1 + \alpha_3 I} - d_2 M\right) + \frac{1}{2} \sigma^2_3 \right] dB_2(t)
+ \left(1 - \frac{1}{V}\right) \left(\beta_4 MI\frac{1}{1 + \alpha_3 I} - d_2 V\right) + \frac{1}{2} \sigma^2_4 \right] dB_3(t)
+ \sigma_2 (I - 1) dB_2(t) + \sigma_3 (M - 1) dB_3(t) + \sigma_3 (V - 1) dB_4(t)
= \mathcal{L} W dt + \sigma_1 (S - 1) dB_1(t) + \sigma_2 (I - 1) dB_2(t) + \sigma_3 (M - 1) dB_3(t)
+ \sigma_4 (V - 1) dB_4(t),
\]
where $\mathcal{L}W : \mathbb{R}_{+}^{4} \rightarrow \mathbb{R}_{+}$ can be written in the following form

$$
\mathcal{L}W(S(t), I(t), M(t), V(t)) = \left[\left(1 - \frac{1}{S}\right)\left(\Lambda_{1} - (\beta_{1} - \frac{\beta_{2}I}{m + I}) \frac{SI}{1 + \alpha_{1}I} - \frac{\beta_{3}SV}{1 + \alpha_{2}V} - d_{1}S\right) + \frac{1}{2} \sigma_{1}^{2}\right]
+ \left(1 - \frac{1}{I}\right)\left(\beta_{1} - \frac{\beta_{2}I}{m + I} \frac{SI}{1 + \alpha_{1}I} + \frac{\beta_{3}SV}{1 + \alpha_{2}V} + (\mu + d_{1} + \gamma)I\right) + \frac{1}{2} \sigma_{2}^{2}
+ \left(1 - \frac{1}{M}\right)\left(\Lambda_{2} - \frac{\beta_{4}MI}{1 + \alpha_{3}I} - d_{2}M\right) + \frac{1}{2} \sigma_{3}^{2} + \left(1 - \frac{1}{V}\right)\left(\frac{\beta_{4}MI}{1 + \alpha_{3}I} - d_{2}V\right) + \frac{1}{2} \sigma_{4}^{2}\right]
$$

$$
= \Lambda_{1} + \Lambda_{2} + 2d_{1} + 2d_{2} + \frac{\beta_{1}}{\alpha_{1}} + \frac{\beta_{3}}{\alpha_{2}} + \frac{\beta_{4}}{\alpha_{3}} + \mu + \gamma + \frac{\beta_{1}}{1 + \alpha_{1}I} + \frac{\beta_{3}V}{1 + \alpha_{2}V} - \frac{\beta_{1}}{m + I} \frac{SI}{1 + \alpha_{1}I} + \frac{\beta_{4}MI}{1 + \alpha_{3}I} - \frac{\beta_{4}MI}{1 + \alpha_{3}I}V
+ \frac{1}{2} \sigma_{1}^{2} + \frac{1}{2} \sigma_{2}^{2} + \frac{1}{2} \sigma_{3}^{2} + \frac{1}{2} \sigma_{4}^{2}
\leq \Lambda_{1} + \Lambda_{2} + 2d_{1} + 2d_{2} + \frac{\beta_{1}}{\alpha_{1}} + \frac{\beta_{3}}{\alpha_{2}} + \frac{\beta_{4}}{\alpha_{3}} + \mu + \gamma + \frac{1}{2} \sigma_{1}^{2} + \frac{1}{2} \sigma_{2}^{2} + \frac{1}{2} \sigma_{3}^{2} + \frac{1}{2} \sigma_{4}^{2} =: \kappa.
$$

Hence, we have

$$
dW(S(t), I(t), M(t), V(t)) \leq \kappa dt + \sigma_{1}(S - 1)dB_{1}(t) + \sigma_{2}(I - 1)dB_{2}(t) + \sigma_{3}(M - 1)dB_{3}(t) + \sigma_{4}(V - 1)dB_{4}(t).
$$

Integrate both sides of Eq. (3.3) from 0 to $\tau_{k} \land T$. It is easy to get that

$$
\int_{0}^{\tau_{k} \land T} dW(S(u), I(u), M(u), V(u)) \leq \kappa du + \int_{0}^{\tau_{k} \land T} \left[\sigma_{1}(S - 1)dB_{1} + \sigma_{2}(I - 1)dB_{2} + \sigma_{3}(M - 1)dB_{3} + \sigma_{4}(V - 1)dB_{4}\right].
$$

Setting $\Omega = \{\tau_{k} \leq T\}$ for $k \geq k_{1}$ and by Eq. (3.1), we get $P(\Omega_{k}) \geq \epsilon$. Further, every $\omega$ from $\Omega$ has at least one of $S(\tau_{k}, \omega), I(\tau_{k}, \omega), M(\tau_{k}, \omega), V(\tau_{k}, \omega)$ equal $k$ or $1/k$. Hence, $W(S(\tau_{k}), I(\tau_{k}), M(\tau_{k}), V(\tau_{k}))$ is not less than $k - 1 - \log k$ or $\frac{1}{k} - 1 + \log k$. That is to say,

$$
W(S(\tau_{k}), I(\tau_{k}), M(\tau_{k}), V(\tau_{k})) \geq (k - 1 - \log k) \land (\frac{1}{k} - 1 - \log \frac{1}{k}).
$$

Combining Eq. (3.4) and Eq. (3.5), we have

$$
W(S(0), I(0), M(0), V(0)) + \kappa(\tau_{k} \land T) \geq E[1_{\Omega_{\omega}}W(S(\tau_{k}), I(\tau_{k}), M(\tau_{k}), V(\tau_{k}))]
\geq \epsilon(k - 1 - \log k) \land (\frac{1}{k} - 1 - \log \frac{1}{k}),
$$

where $1_{\Omega_{\omega}}$ denotes an indicator function of set $\Omega$. Letting $k \rightarrow \infty$ leads to the contradiction

$$
\infty \geq W(S(0), I(0), M(0), V(0)) + \kappa(\tau_{k} \land T) = \infty.
$$

It implies that $\tau_{\varepsilon} = \infty$ a.s. The proof is complete. \qed
Moreover, with any initial value $s_0$, the disease tends to become extinct in the time limit. However, there is no disease-free equilibrium in the stochastic version of the model, which requires other ways to consider its extinction. Define a threshold value

$$
\mathcal{R}_0^S = \frac{1}{\mu_1 + \sigma_2^2/2} \left( \frac{\beta \Lambda_1}{d_1} + \frac{\beta_4 \Lambda_2}{d_2} \right), \quad \sigma_\ast = \min(\sigma_2, \sigma_4).
$$

4. Disease extinction

Theorem 2. Assume $d_1 > \sigma_2^2/2$ and $d_2 > \sigma_3^2/2$. Let $(S(t), I(t), M(t), V(t))$ be the solution of system (1.2) with any initial value $(S(0), I(0), M(0), V(0))$. If $\mathcal{R}_0^S < 1$, then

$$
\limsup_{t \to \infty} \frac{\log(I + V)}{t} \leq \left( \mu_1 + \frac{\sigma_2^2}{2} \right)(\mathcal{R}_0^S - 1) < 0, \quad a.s.
$$

Moreover,

$$
\lim_{t \to \infty} \frac{1}{t} \int_0^t S(u)du = \frac{\Lambda_1}{d_1}, \quad \lim_{t \to \infty} \frac{1}{t} \int_0^t I(u)du = 0 \quad a.s.,
$$

$$
\lim_{t \to \infty} \frac{1}{t} \int_0^t M(u)du = \frac{\Lambda_2}{d_2}, \quad \lim_{t \to \infty} \frac{1}{t} \int_0^t V(u)du = 0 \quad a.s.
$$

Proof. According to Ref. [45], we have

$$
\lim_{t \to \infty} \frac{S(t)}{t} = \lim_{t \to \infty} \frac{I(t)}{t} = \lim_{t \to \infty} \frac{M(t)}{t} = \lim_{t \to \infty} \frac{V(t)}{t} = 0, \quad a.s., \quad (4.1)
$$

$$
\lim_{t \to \infty} \frac{\int_0^t S(u)dB_1(u)}{t} = \lim_{t \to \infty} \frac{\int_0^t I(u)dB_2(u)}{t} = \lim_{t \to \infty} \frac{\int_0^t M(u)dB_3(u)}{t} = \lim_{t \to \infty} \frac{\int_0^t V(u)dB_4(u)}{t} = 0 \quad a.s.. \quad (4.2)
$$

We integrate both sides of the proposed model (1.2) and obtain

$$
\frac{S(t) - S(0)}{t} + \frac{I(t) - I(0)}{t} = \Lambda_1 - \frac{d_1}{t} \int_0^t S(u)du - \frac{\mu + d_1}{t} \int_0^t I(u)du + \frac{\sigma_1}{t} \int_0^t S(u)dB_1(u) + \frac{\sigma_2}{t} \int_0^t I(u)dB_2(u).
$$

It is obvious that

$$
\frac{1}{t} \int_0^t S(u)du = \frac{\Lambda_1}{d_1} - \frac{(d_1 + \mu)}{d_1 t} \int_0^t I(u)du + \frac{\sigma_1}{d_1 t} \int_0^t S(u)dB_1(t) + \frac{\sigma_2}{d_1 t} \int_0^t I(u)dB_2(t) - \frac{S(t) - S(0)}{d_1 t} - \frac{I(t) - I(0)}{d_1 t}. \quad (4.3)
$$

From (4.1) and (4.2), the limit of Eq. (4.3) is

$$
\lim_{t \to \infty} \frac{1}{t} \int_0^t S(u)du = \frac{\Lambda_1}{d_1} - \lim_{t \to \infty} \left( \frac{d_1 + \mu}{d_1 t} \int_0^t I(u)du \right). \quad (4.4)
$$
Similarly, we integrate on both sides of the last two equation of the model (1.2). Hence,

\[
\frac{M(t) - M(0)}{t} + \frac{V(t) - V(0)}{t} = \Lambda_2 - \frac{d_2}{t} \left( \int_0^t M(u)du + \int_0^t V(u)du \right) + \frac{\sigma_3}{t} \int_0^t M(u)dB_3(u) \\
+ \frac{\sigma_4}{t} \int_0^t V(u)dB_4(u).
\]

Combined (4.1) and (4.2), we can get the following equation

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t V(u)du = \frac{\Lambda_2}{d_2} - \lim_{t \to \infty} \frac{1}{t} \int_0^t M(u)du. \tag{4.5}
\]

On the other hand, through the Itô's formula, it follows that

\[
d \log(I + V) = \frac{\beta_4 MI}{(1 + \alpha_1 I)(I + V)} dt + \frac{\beta_3 SV}{(1 + \alpha_2 V)(I + V)} dt + \left( \beta_1 - \frac{\beta_2 I}{m + I} \right) \frac{S I}{(1 + \alpha_1 I)(I + V)} dt \\
- \sigma_2^2 \left( \frac{I}{I + V} \right)^2 dt - \sigma_2^2 \frac{V^2}{(I + V)^2} dt + \frac{\sigma_3 I}{I + V} dB_2(t) + \frac{\sigma_4 V}{I + V} dB_4(t) \\
- (\mu + d_1 + \gamma) \frac{I}{I + V} dt - d_2 \frac{V}{I + V} dt. \\
\leq \frac{\beta_4 MI}{(I + V)} dt + \frac{\beta \left( I + V \right)^2}{(I + V)} dt + \frac{\sigma_3 I}{2 I + V} dB_2(t) + \frac{\sigma_4 V}{2 I + V} dB_4(t) \\
- \mu_1 I + V \frac{dt}{I + V} \sigma_2^2 \frac{(I + V)^2}{2(I + V)^2} dt.
\]

The last term here uses the inequality $2IV \leq (I + V)^2$. Integrate on both sides of the equation and divide it by $t$. Thus,

\[
\frac{1}{t} \log(I + V) \leq \frac{\beta}{t} \int_0^t S(u)du + \frac{\beta_1}{t} \int_0^t M(u)du + \frac{1}{t} \int_0^t \sigma_2 \frac{I}{I + V} dB_2(u) + \frac{1}{t} \int_0^t \sigma_4 \frac{I}{I + V} dB_4(u) \\
- \frac{1}{t} \int_0^t \frac{\sigma_2^2}{2} dt - \frac{1}{t} \int_0^t \mu_1 du,
\]

where $\mu_1 = \min(\mu + d_1 + \gamma, d_2), \sigma_* = \min(\sigma_2, \sigma_4)$, and $\beta = \max(\beta_1, \beta_3)$. From (4.4) and (4.5), we can get

\[
\frac{1}{t} \log(I + V) \leq \beta_1 \left( \frac{\Lambda_1}{d_1} - \frac{(d_1 + \mu)}{d_1} \int_0^t I(u)du \right) + \beta_4 \left( \frac{\Lambda_2}{d_2} - \frac{1}{t} \int_0^t V(u)du \right) \\
+ \frac{1}{t} \int_0^t \sigma_2 \frac{I}{I + V} dB_2(u) + \frac{1}{t} \int_0^t \sigma_4 \frac{I}{I + V} dB_4(u) - \frac{1}{t} \int_0^t \frac{\sigma_2^2}{2} dt - \frac{1}{t} \int_0^t \mu_1 du. \tag{4.6}
\]

According to Lemma 1, it is obtained that

\[
\limsup_{t \to \infty} \left( \frac{1}{t} \int_0^t \sigma_4 \frac{V}{I + V} dB_4(u) + \frac{1}{t} \int_0^t \sigma_2 \frac{I}{I + V} dB_2(u) \right) = 0, \quad a.s.. \tag{4.7}
\]

Through (4.6) and (4.7), we have

\[
\limsup_{t \to \infty} \frac{\log(I + V)}{t} \leq \left( \mu_1 + \frac{\sigma_*^2}{2} \right) (\mathcal{R}^S_0 - 1) < 0, \quad a.s..
\]
It means that \( \lim_{t \to \infty} \frac{1}{I} \int_0^t I(u) du = 0 \), and \( \lim_{t \to \infty} \frac{1}{V} \int_0^t V(u) du = 0 \), a.s.. Combining (4.4) and (4.5), it is obvious that

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t S(u) du = \frac{\Lambda_1}{d_1}, \quad \lim_{t \to \infty} \frac{1}{t} \int_0^t M(u) du = \frac{\Lambda_2}{d_2}, \quad \text{a.s.}
\]

The proof is complete. \( \square \)

5. Persistence in the mean of the disease

The most interesting aspect in the study of epidemic modeling is the extinction and persistence of epidemic; in the previous section we studied disease extinction and in this section we will show that diseases are persistent in the mean.

Theorem 3. Assume \( d_1 > \frac{\sigma_1^2 \sigma_2^2}{2} \) and \( d_2 > \frac{\sigma_1^2 \sigma_3^2}{2} \). If

\[
\mathcal{R}_1^c = \frac{9 \sqrt{\Lambda_1^2 \Lambda_2^2 d_1 d_2 (\beta_1 - \beta_2) \beta_3 \beta_4 + \frac{1}{2} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)}}{4d_1 + 3d_2 + \mu + \gamma} > 1,
\]

then for any given initial value \((S(0), I(0), M(0), V(0)) \in \mathbb{R}_+^4\), the solution of system (1.2) has the following properties

(i) \( \liminf_{t \to \infty} \frac{1}{t} \int_0^t S(u) du \geq \frac{\Lambda_1}{d_1 + \beta_1/\alpha_1 + \beta_2/\alpha_2}, \quad \text{a.s.} \)

(ii) \( \liminf_{t \to \infty} \frac{1}{t} \int_0^t M(u) du \geq \frac{\Lambda_2}{d_2 + \beta_3/\alpha_3}, \quad \text{a.s.} \)

(iii) \( \liminf_{t \to \infty} \frac{1}{t} \int_0^t I(u) du + \liminf_{t \to \infty} \frac{1}{t} \int_0^t V(u) du \geq \frac{4d_1 + 3d_2 + \mu + \gamma}{(\beta_1 + \beta_4 + d_1/\alpha_3 + d_1/\alpha_1) \wedge (\beta_3 + d_2/\alpha_2)} (\mathcal{R}_1^c - 1), \quad \text{a.s.} \)

Proof. (i) From the first equation of system (1.2) integrating the above inequality and dividing both sides by \( t \), we get

\[
\frac{S(t) - S(0)}{t} = \Lambda_1 - \frac{1}{t} \int_0^t \left( \frac{\beta_1 - \beta_2 I(t)}{I(t) + m} \right) S(u) I(u) \left[ \frac{1}{1 + \alpha_1 I(u)} - \frac{1}{1 + \alpha_2 V(u)} \right] du - \frac{1}{t} \int_0^t d_1 S(u) du - \frac{\sigma_1}{t} \int_0^t S(u) dB_1(u).
\]

In view of Theorem 1, for any initial value \((S(0), I(0), M(0), V(0)) \in \mathbb{R}_+^4\), there is a unique global solution \((S(t), I(t), M(t), V(t)) \in \mathbb{R}_+^4\). Thus,

\[
\frac{S(t) - S(0)}{t} + \frac{\sigma_1}{t} \int_0^t S(u) dB_1(u) \geq \Lambda_1 - \frac{1}{t} \int_0^t \frac{\beta_1 S(u)}{\alpha_1} du - \frac{1}{t} \int_0^t \frac{\beta_2 S(u)}{\alpha_2} du - \frac{1}{t} \int_0^t d_1 S(u) du.
\]

(5.1)

Through the strong law of large numbers for local martingales, we have

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t S(u) dB_1(u) = 0 \quad \text{a.s.,}
\]
which together with (5.1) yields

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t S(u) du \geq \frac{\Lambda_1}{d_1 + \beta_1/\alpha_1 + \beta_2/\alpha_2} \quad \text{a.s.,}
\]

This is the required assertion (i).

(ii) From the third equation of system (1.2) integrating the above inequality and dividing both sides by \(t\), we get

\[
\frac{M(t) - M(0)}{t} = \Lambda_2 - \frac{1}{t} \int_0^t \frac{\beta_4 M(u) I(u)}{1 + \alpha_3 I(u)} du - \frac{1}{t} \int_0^t d_2 M(u) du - \frac{\sigma_3}{t} \int_0^t M(u) dB_3(u).
\]

Then

\[
\frac{M(t) - M(0)}{t} + \frac{\sigma_3}{t} \int_0^t M(u) dB_3(u) \geq \Lambda_2 - \frac{1}{t} \int_0^t \frac{\beta_4 M(u)}{\alpha_3} du - \frac{1}{t} \int_0^t d_2 M(u) du.
\]  

(5.2)

According to the strong law of large numbers of local martingales, we have

\[
\lim_{t \to \infty} \left( \frac{M(t) - M(0)}{t} + \frac{\sigma_3}{t} \int_0^t M(u) dB_3(u) \right) = 0 \quad \text{a.s.,}
\]

which together with (5.2) yields

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t M(u) du \geq \frac{\Lambda_2}{d_2 + \beta_4/\alpha_3} \quad \text{a.s.}
\]

This is the required assertion (ii).

(iii) First, define a function \(W_2(S, I, V) = -\ln S - \ln I - \ln M - \ln V\). According to the Itô’s formula:

\[
dW_2(t) = \mathcal{L}W_2(t) dt - \sigma_1 \ dB_1(t) - \sigma_2 \ dB_2(t) - \sigma_3 \ dB_3(t) - \sigma_4 \ dB_4(t),
\]
Thus, the proof is complete.

According to the powerful number law of a martingale, we integrate the above inequality in the interval $[0, t]$, divide it by $t$, and take the limit to $t$. Thus,

$$
0 \leq -9 \sqrt{\frac{\Lambda_2}{\Lambda_3}} \int_0^t \sigma_1 \, dB_1(u) - \lim_{t \to \infty} \frac{1}{t} \int_0^t \sigma_2 \, dB_2(u) - \lim_{t \to \infty} \frac{1}{t} \int_0^t \sigma_3 \, dB_3(u) - \lim_{t \to \infty} \frac{1}{t} \int_0^t \sigma_4 \, dB_4(u).
$$

According to the powerful number law of a martingale,

$$
\lim_{t \to \infty} \frac{1}{t} \int_0^t \sigma_1 \, dB_1(u) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \sigma_2 \, dB_2(u) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \sigma_3 \, dB_3(u) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \sigma_4 \, dB_4(u) = 0.
$$

It follows that (5.3) becomes

$$
\lim_{t \to \infty} \frac{1}{t} \int_0^t I(u) \, du + \lim_{t \to \infty} \frac{1}{t} \int_0^t V(u) \, du \geq 9 \sqrt{\frac{\Lambda_2^2}{\Lambda_3}} \int_0^t (\beta_1 - \beta_2) \beta_3 \beta_4 - (4d_1 + 3d_2 + \mu + \gamma) + \frac{1}{2} \left( \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 \right) \text{ a.s.}
$$

The proof is complete.
6. Stationary distribution

The ergodic property for an epidemic model means that the stochastic model has a unique stationary distribution that forecasts the permanence of the epidemic in the future. That means the disease persists for all time regardless of the initial condition.

In this section, we provide a sufficient condition for the existence of a stationary distribution in the model (1.2). Denote

where $r_1 = \frac{1}{2} \sum_{i=1}^{4} \sigma_i^2 + 2d_1 + 3d_2 + \mu + \gamma + \frac{\beta_1}{a_1} + \frac{\beta_3}{a_2} + \frac{\beta_4}{a_3}$, and $r_2 = \frac{1}{2} \sum_{i=1}^{4} \sigma_i^2 + 4d_1 + 2d_2 + \mu + \gamma + \frac{\beta_1}{a_1} + \frac{\beta_3}{a_2} + \frac{\beta_4}{a_3}$. Theorem 4. If $\mathcal{R}_2^S > 1$, then model (1.2) has a unique stationary distribution $\pi(\cdot)$ with ergodicity.

Proof. The diffusion matrix for model (1.2) is

$$
\mathcal{A} = \begin{pmatrix} 
\sigma_1^2 S^2 & 0 & 0 & 0 \\
0 & \sigma_2^2 I^2 & 0 & 0 \\
0 & 0 & \sigma_3^2 M^2 & 0 \\
0 & 0 & 0 & \sigma_4^2 V^2 
\end{pmatrix}.
$$

Denote $M = \min_{(S,I,M,V)\in\mathbb{R}^4} \left\{ \sigma_1^2 S^2, \sigma_2^2 I^2, \sigma_3^2 M^2, \sigma_4^2 V^2 \right\}$. It follows that

$$
\sum_{i,j=1}^{4} a_{ij}(S,I,M,V)\xi_i\xi_j = \sigma_1^2 S^2 \xi_1^2 + \sigma_2^2 I^2 \xi_2^2 + \sigma_3^2 M^2 \xi_3^2 + \sigma_4^2 V^2 \xi_4^2 \geq M|\xi|^2
$$

for $(S,I,M,V) \in D, \xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4$, where $D = \left( \frac{1}{k}, k \right) \times \left( \frac{1}{k}, k \right) \times \left( \frac{1}{k}, k \right)$ and $k$ is a sufficiently large integer. Therefore, the condition (i) in Lemma 2 is satisfied. Next, we prove the condition (ii) in Lemma 2. Let

$$
\mathcal{Y}_1 = - \log S - \log M - \log V - \log I + \alpha_2 d_2 V, \\
\mathcal{Y}_2 = - \log S - \log M - \log V + (\alpha_1 + \alpha_3)(S + I), \\
\mathcal{Y}_3 = - \log S - \log M, \\
\mathcal{Y}_4 = \frac{1}{1 + \theta}(S + I + M + V)^{\theta + 1}.
$$

Denote $\lambda_i = r_i(\mathcal{R}_i - 1)(i = 1, 2)$, $\sigma^2 = \sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2$, $b = 2d_1 + \mu + \gamma + \frac{\beta_1}{a_1} + \frac{\beta_3}{a_2} + \frac{\beta_4}{a_3} + \frac{\sigma_1^2 + \sigma_2^2}{2}$ and $d = d_1 \wedge d_2$. We construct a $C^2$-function $\mathcal{F} : \mathbb{R}^4 \rightarrow \mathbb{R}$ as follows

$$
\mathcal{F}(S,I,M,V) = \Theta_1 \mathcal{Y}_1 + \Theta_2 \mathcal{Y}_2 + \mathcal{Y}_3 + \mathcal{Y}_4,
$$

where $\Theta_i(i = 1, 2)$ are sufficiently large positive constants, satisfying $-\Theta_1 \lambda_1 + F_2 \leq -3, -\Theta_2 \lambda_2 + F_3 \leq$
Here, $\theta$ is a positive constant satisfying $d > (\theta + 1)\sigma^2/2$. It means that

$$\liminf_{k \to \infty, (S, I, M, V) \in \mathbb{R}^4_+ \setminus D} \tilde{\nu}(S, I, M, V) = \infty,$$

and $\tilde{\nu}(S, I, M, V)$ is a continuous function. Then function $\tilde{\nu}(S, I, M, V)$ must have a minimum point $(\bar{S}, \bar{I}, \bar{M}, \bar{V}) \in \mathbb{R}^4_+$. Further, we construct a nonnegative $C^2$ function $\nu: \mathbb{R}^4_+ \to \mathbb{R}$ in the following form

$$\nu(S, I, M, V) = \Theta_1 \gamma_1 + \Theta_2 \gamma_2 + \gamma_3 + \gamma_4 - \tilde{\nu}(\bar{S}, \bar{I}, \bar{M}, \bar{V}).$$

Applying Itô's formula to $\gamma_1$, we get

$$L\gamma_1 = -\frac{1}{S} \left[ \Lambda_1 - d_1 S - \left( \beta_1 - \beta_2 \frac{I}{I + m} \right) \frac{SI}{1 + \alpha_1 I} - \beta_3 \frac{SV}{1 + \alpha_2 V} \right] + \frac{\sigma_1^2}{2} - \frac{1}{I} \left[ -(d_1 + \mu + \gamma) I - \left( \beta_1 - \beta_2 \frac{I}{I + m} \right) \frac{SI}{1 + \alpha_1 I} + \beta_3 \frac{SV}{1 + \alpha_2 V} \right] + \frac{\sigma_2^2}{2} - \frac{1}{M} \left[ \Lambda_1 - \beta_4 \frac{MI}{1 + \alpha_3 I} - d_2 M \right] + \frac{\sigma_3^2}{2} - \frac{1}{V} \left[ \beta_4 \frac{MI}{1 + \alpha_3 I} - d_2 V \right] + \frac{\sigma_4^2}{2} - \frac{\alpha_2 d_2 \beta_4 MI}{(1 + \alpha_3 I)} - d_2 \alpha_2 V \leq -\frac{\Lambda_1}{2S} - \frac{\Lambda_2}{M} - \left( \beta_1 - \beta_2 \right) \frac{S}{1 + \alpha_1 I} - \frac{\Lambda_1}{2S} - \frac{\beta_3 SV}{(1 + \alpha_2 V) I} + \frac{\beta_4 MI}{(1 + \alpha_3 I) V} + d_2(1 + \alpha_2 V).$$

\[ \begin{align*}
F_2 &= \sup_{(S, I, M, V) \in \mathbb{R}^4_+} \left\{ \frac{1}{2} \left( \alpha_2 d_2 \beta_4 I^2 \right) S^2 - \frac{1}{2} \left( d - \frac{(\theta + 1)\sigma^2}{2} \right) \left( S^\theta + I^\theta + M^\theta + V^\theta \right) + b + B \right\}, \\
F_3 &= \sup_{(S, I, M, V) \in \mathbb{R}^4_+} \left\{ \Theta_1 \alpha_2 d_2 \beta_4 MI - \frac{1}{2} \left( d - \frac{(\theta + 1)\sigma^2}{2} \right) \left( S^\theta + I^\theta + M^\theta + V^\theta \right) + b + B \right\}, \\
B &= \sup_{(S, I, M, V) \in \mathbb{R}^4_+} \left\{ (\Lambda_1 + \Lambda_2) (S + I + M + V)^{\theta + 1} - \frac{1}{2} \left( d - \frac{(\theta + 1)\sigma^2}{2} \right) \left( S + I + M + V \right)^{\theta + 2} \right\} < \infty,
\end{align*} \]
\[
\mathcal{L} \mathcal{V}_2 = -\frac{1}{S} \left[ \Lambda_1 - d_1 S - \left( \beta_1 - \beta_2 \frac{I}{1 + \alpha_1 I} \right) \frac{S I}{1 + \alpha_1 I} - \beta_3 \frac{S V}{1 + \alpha_2 V} \right] + \frac{\sigma_1^2}{2} \\
- \frac{1}{I} \left[ -(d_1 + \mu + \gamma) I - \left( \beta_1 - \beta_2 \frac{I}{1 + \alpha_1 I} \right) \frac{S I}{1 + \alpha_1 I} + \beta_3 \frac{S V}{1 + \alpha_2 V} \right] + \frac{\sigma_2^2}{2} \\
- \frac{1}{M} \left[ \Lambda_1 - \beta_4 \frac{M I}{1 + \alpha_3 I} - d_2 M \right] + \frac{\sigma_4^2}{2} - \frac{1}{V} \left[ \beta_4 \frac{M I}{1 + \alpha_3 I} - d_2 V \right] + \frac{\sigma_4^2}{2} \\
- d_1 (\alpha_1 + \alpha_3) I - d_1 \alpha_1 (\alpha_1 + \alpha_3) S - (\mu + \gamma)(\alpha_1 + \alpha_3) I \\
\leq -\frac{\Lambda_1}{2S} - \frac{\Lambda_2}{M} - (\beta_1 - \beta_2) \frac{S}{1 + \alpha_1 I} - \frac{\Lambda_1}{2S} - \frac{\beta_3 S V}{(1 + \alpha_2 V)I} - \frac{\beta_4 M I}{(1 + \alpha_3 I) V} - d_1 (1 + \alpha_1 I) \\
- d_1 (1 + \alpha_3 I) + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{\sigma_4^2}{2} + 4d_1 + 2d_2 + (\mu + \gamma) + \frac{\beta_1}{\alpha_1} + \frac{\beta_3}{\alpha_2} + \frac{\beta_4}{\alpha_3} \\
\leq -8 \sqrt{\frac{(\beta_1 - \beta_2) \beta_3 \beta_4 \Lambda_1^2 \Lambda_2}{4(1 + \alpha_3 V)}} + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{\sigma_4^2}{2} + 4d_1 + 2d_2 + (\mu + \gamma) \\
+ \frac{\beta_1}{\alpha_1} + \frac{\beta_3}{\alpha_2} + \frac{\beta_4}{\alpha_3},
\]

\[
\mathcal{L} \mathcal{V}_3 = -\frac{1}{S} \left[ \Lambda_1 - d_1 S - \left( \beta_1 - \beta_2 \frac{I}{1 + \alpha_1 I} \right) \frac{S I}{1 + \alpha_1 I} - \beta_3 \frac{S V}{1 + \alpha_2 V} \right] + \frac{\sigma_1^2}{2} \\
- \frac{1}{M} \left[ \Lambda_2 - \beta_4 \frac{M I}{1 + \alpha_3 I} - d_2 M \right] + \frac{\sigma_4^2}{2} \\
\leq -\frac{\Lambda_1}{S} - \frac{\Lambda_2}{M} + 2d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_3}{\alpha_2} + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2},
\]

\[
\mathcal{L} \mathcal{V}_4 = (S + I + M + V)^{\theta + 1} \left[ \Lambda_1 - d_1 (S + I + R) - (\mu + \gamma) I + \Lambda_2 - d_2 (M + V) \right] \\
+ (\theta + 1) (S + I + M + V)^{\theta} \left[ \frac{\sigma_1^2}{2} S^2 + \frac{\sigma_2^2}{2} I^2 + \frac{\sigma_3^2}{2} M^2 + \frac{\sigma_4^2}{2} V^2 \right] \\
\leq (S + I + M + V)^{\theta + 1} (\Lambda_1 + \Lambda_2) - (S + I + R + M + V)^{\theta + 2} (d_1 \wedge d_2) \\
+ \frac{1}{2} (\theta + 1) (S + I + M + V)^{\theta} \sigma_2^2 (S + I + M + V)^2 \\
\leq (S + I + M + V)^{\theta + 1} (\Lambda_1 + \Lambda_2) - \left( d - \frac{(\theta + 1) \sigma_2^2}{2} \right) (S + I + M + V)^{\theta + 2} \\
\leq B - \frac{1}{2} \left( d - \frac{(\theta + 1) \sigma_2^2}{2} \right) \left( S^{\theta + 2} + I^{\theta + 2} + M^{\theta + 2} + V^{\theta + 2} \right),
\]
where

\[ B = \sup_{(S, I, M, V) \in \mathbb{R}_+^4} \left\{ (\Lambda_1 + \Lambda_2)(S + I + M + V)^{\theta+1} - \frac{1}{2} \left( d - \frac{(\theta + 1)\sigma^2}{2} \right) (S + I + M + V)^{\theta+2} \right\} < \infty. \]

Thus, it follows that

\[ \mathcal{L} \psi \leq - \Theta_1 r_1(\mathcal{R}^S_3 - 1) - \Theta_2 r_2(\mathcal{R}^S_4 - 1) + 2d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_3}{\alpha_2} + \frac{\beta_4}{\alpha_3} + \frac{\sigma^2_1 + \sigma^2_3}{2} \]

\[ + \Theta_1 (\alpha_2 d_2 \beta_4) M I + B - \frac{1}{2} \left( d - \frac{(\theta + 1)\sigma^2}{2} \right) (S^{\theta+2} + I^{\theta+2} + M^{\theta+2} + V^{\theta+2}) - \frac{\Lambda_1}{S} - \frac{\Lambda_2}{M}. \]

A closed subset is defined as

\[ D = \left\{ (S, I, M, V) \in \mathbb{R}_+^4 : \epsilon \leq S \leq \frac{1}{\epsilon}, \epsilon \leq I \leq \frac{1}{\epsilon}, \epsilon \leq M \leq \frac{1}{\epsilon}, \epsilon \leq V \leq \frac{1}{\epsilon} \right\}, \]

where \( \epsilon > 0 \) is sufficiently small constants satisfying the following conditions

\[ - \frac{\Lambda_1}{\epsilon} + F_3 < 1, \quad (6.1) \]

\[ - \Theta_1 r_3(7 \sqrt{\frac{\beta_1 - \beta_2}{4(1 + \alpha \epsilon)(1 + \alpha \epsilon)} - 1} + F_2 + \frac{\Theta_3^2 \epsilon^2}{2} < -2, \quad (6.2) \]

\[ H + \Theta_1 (\alpha_2 d_2 \beta_4) \frac{1}{\epsilon^2} - \frac{1}{2} \left( d - \frac{(\theta + 1)\sigma^2}{2} \right) (\frac{1}{\epsilon})^{\theta+2} < -1, \quad (6.3) \]

\[ - \frac{\Lambda_2}{\epsilon} + F_2 < -1, \quad (6.4) \]

\[ - \Theta_2 r_2(8 \sqrt{\frac{\beta_1 - \beta_2}{4(1 + \alpha \epsilon)} - 1}) + F_3 < -1, \quad (6.5) \]

\[ H + \Theta_1 (\alpha_2 d_2 \beta_4) \frac{1}{\epsilon^2} - \frac{1}{2} \left( d - \frac{(\theta + 1)\sigma^2}{2} \right) (\frac{1}{\epsilon})^{\theta+2} < -1, \quad (6.6) \]

where \( H \) is a constant and is determined later. Denote \( Y = (S, I, M, V) \). We divide \( \mathbb{R}_+^4 \setminus D \) into the following eight cases

\[ D_1 = \{ Y \in \mathbb{R}_+^4, 0 < S < \epsilon \}, \quad D_2 = \{ Y \in \mathbb{R}_+^4, 0 < I < \epsilon \}, \]

\[ D_3 = \{ Y \in \mathbb{R}_+^4, S > \frac{1}{\epsilon}, \epsilon < M < \frac{1}{\epsilon}, \epsilon < I < \frac{1}{\epsilon} \}, \quad D_4 = \{ Y \in \mathbb{R}_+^4, I > \frac{1}{\epsilon}, \epsilon < M < \frac{1}{\epsilon} \}, \]

\[ D_5 = \{ Y \in \mathbb{R}_+^4, 0 < M < \epsilon \}, \quad D_6 = \{ Y \in \mathbb{R}_+^4, 0 < V < \epsilon \}, \]

\[ D_7 = \{ Y \in \mathbb{R}_+^4, \frac{1}{\epsilon} < M, \epsilon < I < \frac{1}{\epsilon} \}, \quad D_8 = \{ Y \in \mathbb{R}_+^4, \frac{1}{\epsilon} < V, \epsilon < M < \frac{1}{\epsilon}, \epsilon < I < \frac{1}{\epsilon} \}. \]
Now, we will prove that $\mathcal{L}V(S, I, M, V) < -1$ on $\mathbb{R}^4 \setminus D$; this is equivalent to proving that it is valid on the above eight subsets.

Case 1. When $(S, I, M, V) \in D_1$, we get
\[
\mathcal{L} \mathcal{V} \leq 2d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_4}{\alpha_4} + \frac{\beta_3}{\alpha_3} + \frac{\sigma_1^2 + \sigma_2^2}{2} + \Theta_1 (\alpha_2 d_2 \beta_4) M I
\]
\[+ B - \frac{1}{2} \left( d - \frac{(\theta + 1)\sigma^2}{2} \right) (S^{\theta + 2} + I^{\theta + 2} + M^{\theta + 2} + V^{\theta + 2}) - \frac{\Lambda_1}{S} \]
\[\leq - \frac{\Lambda_1}{S} + F_3 \leq - \frac{\Lambda_1}{\epsilon} + F_3, \]
where
\[
F_3 = \sup_{(S, I, M, V) \in \mathbb{R}^4_+} \left\{ \Theta_1 \alpha_2 d_2 \beta_4 M I - \frac{1}{2} \left( d - \frac{(\theta + 1)\sigma^2}{2} \right) (S^{\theta + 2} + I^{\theta + 2} + M^{\theta + 2} + V^{\theta + 2}) + a + B \right\}. \]

According to (6.1), we have $\mathcal{L} \mathcal{V}(S, I, M, V) < -1$ for any $(S, I, M, V) \in D_1$.

Case 2. When $(S, I, M, V) \in D_2$, we have
\[
\mathcal{L} \mathcal{V} \leq - \Theta_1 r_3 (7 \sqrt{\frac{(\beta_1 - \beta_2) \beta_3 \beta_4 \Lambda_1^2 \Lambda_2}{4(1 + \alpha_1 I)(1 + \alpha_3 I)} - 1} + 2d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_3}{\alpha_3} + \frac{\sigma_1^2 + \sigma_2^2}{2}
\]
\[+ \frac{(\alpha_2 d_2 \beta_4) I}{2} + B - \frac{1}{2} \left( d - \frac{(\theta + 1)\sigma^2}{2} \right) (S^{\theta + 2} + I^{\theta + 2} + M^{\theta + 2} + V^{\theta + 2}) + \frac{\Theta_1^2 I^2}{2}
\]
\[\leq - \Theta_1 r_3 (7 \sqrt{\frac{(\beta_1 - \beta_2) \beta_3 \beta_4 \Lambda_1^2 \Lambda_2}{4(1 + \alpha_1 I)(1 + \alpha_3 I)} - 1} + 2d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_3}{\alpha_3} + \frac{\sigma_1^2 + \sigma_2^2}{2}
\]
\[+ \frac{(\alpha_2 d_2 \beta_4) I}{2} + B - \frac{1}{2} \left( d - \frac{(\theta + 1)\sigma^2}{2} \right) (S^{\theta + 2} + I^{\theta + 2} + M^{\theta + 2} + V^{\theta + 2}) + \frac{\Theta_1^2 I^2}{2}
\]
\leq - \Theta_1 r_3 (7 \sqrt{\frac{(\beta_1 - \beta_2) \beta_3 \beta_4 \Lambda_1^2 \Lambda_2}{4(1 + \alpha_1 I)(1 + \alpha_3 I)} - 1} + F_2 + \frac{\Theta_1^2 I^2}{2},
\]
where
\[
F_2 = \sup_{(S, I, M, V) \in \mathbb{R}^4_+} \left\{ \frac{1}{2} \frac{(\alpha_2 d_2 \beta_4) I}{2} + \frac{1}{2} \left( d - \frac{(\theta + 1)\sigma^2}{2} \right) (S^{\theta + 2} + I^{\theta + 2} + M^{\theta + 2} + V^{\theta + 2}) + a + B \right\}. \]

Based on (6.2) we get $\mathcal{L} \mathcal{V}(S, I, M, V) < -1$ for any $(S, I, M, V) \in D_2$.

Case 3. When $(S, I, M, V) \in D_3$, it yields that
\[
\mathcal{L} \mathcal{V} \leq d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_4}{\alpha_4} + \frac{\beta_3}{\alpha_3} + \frac{\sigma_1^2 + \sigma_2^2}{2} + \Theta_1 (\alpha_2 d_2 \beta_4) M I
\]
\[+ B - \frac{1}{2} \left( d - \frac{(\theta + 1)\sigma^2}{2} \right) (S^{\theta + 2} + I^{\theta + 2} + M^{\theta + 2} + V^{\theta + 2})
\]
\[\leq H + \Theta_1 (\alpha_2 d_2 \beta_4) \frac{1}{e^2} - \frac{1}{2} \left( d - \frac{(\theta + 1)\sigma^2}{2} \right) \left( \frac{1}{e} \right)^{\theta + 2}
\]
\[\leq H + \Theta_1 (\alpha_2 d_2 \beta_4) \frac{1}{e^2} - \frac{1}{2} \left( d - \frac{(\theta + 1)\sigma^2}{2} \right) \left( \frac{1}{e} \right)^{\theta + 2},
\]
where \( H = d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_4}{\alpha_3} + \frac{\beta_3 + \alpha^2}{\alpha_3} + \frac{\sigma^2 + \alpha_3^2}{2} + B \), and from (6.3), we have that \( \mathcal{L} \mathcal{V}(S, I, M, V) < -1 \) for any \((S, I, M, V) \in D_3 \).

Case 4. When \((S, I, M, V) \in D_4\), then

\[
\mathcal{L} \mathcal{V} \leq 2d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_4}{\alpha_3} + \frac{\beta_3 + \alpha_3^2}{2} + \Theta_1(\alpha_2d_3\beta_4)MI + B - \frac{1}{2} \left( d - \frac{(\theta + 1) \sigma_1^2}{2} \right) (S^{\theta+2} + I^{\theta+2} + M^{\theta+2} + V^{\theta+2}) \leq H + \Theta_1(\alpha_2d_3\beta_4) \left( \frac{1}{\epsilon^2} \right) - \frac{1}{2} \left( d - \frac{(\theta + 1) \sigma_1^2}{2} \right) \left( \frac{1}{\epsilon} \right)^{\theta+2}.
\]

Again from (6.3) we find that \( \mathcal{L} \mathcal{V}(S, I, M, V) < -1 \) for any \((S, I, M, V) \in D_4 \).

Case 5. When \((S, I, M, V) \in D_5\), we have

\[
\mathcal{L} \mathcal{V} \leq 2d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_4}{\alpha_3} + \frac{\beta_3 + \alpha_3^2}{2} + \Theta_1(\alpha_2d_3\beta_4)MI + B - \frac{1}{2} \left( d - \frac{(\theta + 1) \sigma_1^2}{2} \right) (S^{\theta+2} + I^{\theta+2} + M^{\theta+2} + V^{\theta+2}) \leq - \frac{\Lambda_2}{M} + F_3 \leq - \frac{\Lambda_1}{\epsilon} + F_3,
\]

By means of (6.4) we obtain \( \mathcal{L} \mathcal{V}(S, I, M, V) < -1 \) for any \((S, I, M, V) \in D_5 \).

Case 6. When \((S, I, M, V) \in D_6\), we can get

\[
\mathcal{L} \mathcal{V} \leq - \Theta_2r_2(\delta \sqrt{\frac{(\beta_1 - \beta_2)\beta_3\beta_4\Lambda^2_1\Lambda_2}{4(1 + \alpha_3V)} - 1}) + 2d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_3}{\alpha_2} + \frac{\beta_4}{\alpha_3} + \frac{\sigma^2 + \alpha_3^2}{2} + \Theta(\alpha_2d_3\beta_4)MI + B - \frac{1}{2} \left( d - \frac{(\theta + 1) \sigma_1^2}{2} \right) (S^{\theta+2} + I^{\theta+2} + M^{\theta+2} + V^{\theta+2}) \leq - \Theta_2r_2(\delta \sqrt{\frac{(\beta_1 - \beta_2)\beta_3\beta_4\Lambda^2_1\Lambda_2}{4(1 + \alpha_3V)} - 1}) + 2d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_3}{\alpha_2} + \frac{\beta_4}{\alpha_3} + \frac{\sigma^2 + \alpha_3^2}{2} + \Theta(\alpha_2d_3\beta_4)MI + B - \frac{1}{2} \left( d - \frac{(\theta + 1) \sigma_1^2}{2} \right) (S^{\theta+2} + I^{\theta+2} + M^{\theta+2} + V^{\theta+2}) \leq - \Theta_2r_2(\delta \sqrt{\frac{(\beta_1 - \beta_2)\beta_3\beta_4\Lambda^2_1\Lambda_2}{4(1 + \alpha_3V)} - 1}) + F_3.
\]

By (6.5), we have \( \mathcal{L} \mathcal{V}(S, I, M, V) < -1 \) for any \((S, I, M, V) \in D_6 \).
Case 7. When \((S, I, M, V) \in D_7\), it follows that
\[
\mathcal{L} \nu \leq 2d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_4}{\alpha_3} + \frac{\beta_3}{\alpha_2} + \frac{\sigma_1^2 + \sigma_3^2}{2} + \Theta_1 (\alpha_2 d_2 \beta_4) MI
+ B - \frac{1}{2} \left( d - \frac{(\theta + 1)\sigma^2}{2} \right) \left( S^{\theta+2} + I^{\theta+2} + M^{\theta+2} + V^{\theta+2} \right)
\leq d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_4}{\alpha_3} + \frac{\beta_3}{\alpha_2} + \frac{\sigma_1^2 + \sigma_3^2}{2} + \Theta_1 (\alpha_2 d_2 \beta_4) \frac{1}{\epsilon^2}
+ B - \frac{1}{2} \left( d - \frac{(\theta + 1)\sigma^2}{2} \right) \left( \frac{1}{\epsilon} \right)^{\theta+2}
\leq H + \Theta_1 (\alpha_2 d_2 \beta_4) \frac{1}{\epsilon^2} - \frac{1}{2} \left( d - \frac{(\theta + 1)\sigma^2}{2} \right) \left( \frac{1}{\epsilon} \right)^{\theta+2}.
\]
Using (6.6) we have \(\mathcal{L} \nu (S, I, M, V) < -1\) for any \((S, I, M, V) \in D_7\).

Case 8. When \((S, I, M, V) \in D_8\), then
\[
\mathcal{L} \nu \leq 2d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_4}{\alpha_3} + \frac{\beta_3}{\alpha_2} + \frac{\sigma_1^2 + \sigma_3^2}{2} + \Theta_1 (\alpha_2 d_2 \beta_4) MI
+ B - \frac{1}{2} \left( d - \frac{(\theta + 1)\sigma^2}{2} \right) \left( S^{\theta+2} + I^{\theta+2} + M^{\theta+2} + V^{\theta+2} \right)
\leq 2d_1 + \mu + \gamma + \frac{\beta_1}{\alpha_1} + \frac{\beta_4}{\alpha_3} + \frac{\beta_3}{\alpha_2} + \frac{\sigma_1^2 + \sigma_3^2}{2} + \Theta_1 (\alpha_2 d_2 \beta_4) \frac{1}{\epsilon^2}
+ B - \frac{1}{2} \left( d - \frac{(\theta + 1)\sigma^2}{2} \right) \left( \frac{1}{\epsilon} \right)^{\theta+2}
\leq H + \Theta_1 (\alpha_2 d_2 \beta_4) \frac{1}{\epsilon^2} - \frac{1}{2} \left( d - \frac{(\theta + 1)\sigma^2}{2} \right) \left( \frac{1}{\epsilon} \right)^{\theta+2}.
\]
Again using (6.6), we have \(\mathcal{L} \nu (S, I, M, V) < -1\) for any \((S, I, M, V) \in D_8\).

In Cases 1-8, we choose enough small value \(\epsilon\) such that \(\mathcal{L} \nu (S, I, M, V) < -1\) for any \((S, I, M, V) \in D_i (i = 1, 2, \ldots, 8)\). Thus, \(\mathcal{L} \nu (S, I, M, V) < -1\) for all \((S, I, M, V) \in \mathbb{R}_+^4 \setminus D\). On the other hand,
\[
d\nu (S, I, M, V) < -dt + \sigma_1 \left[ S^2 + \Theta_2 (\alpha_1 + \alpha_3) S + SI + SM + SV - (\Theta_1 + \Theta_2 + 1) \right] dB_1(t)
+ \sigma_2 \left[ I^2 + IS + \Theta_2 (\alpha_1 + \alpha_3) I + IM + IV - \Theta_1 \right] dB_2(t)
+ \sigma_3 \left[ M^2 + MS + MI + MV - (\Theta_1 + \Theta_2 + 1) \right] dB_3(t)
+ \sigma_4 \left[ V^2 + VS + VI + VM + (\Theta_1 \alpha_2 d_2) V - (\Theta_1 + \Theta_2) \right] dB_4(t).
\]
Assume that \((S(0), I(0), M(0), V(0)) = (y_1, y_2, y_3, y_4) = y \in \mathbb{R}_+^4 \setminus D\), and \(\tau^y\) is that time at which a path starting from \(y\) reach to the set \(D\),
\[
\tau_n = \inf \{ t : |y(t)| = n \} \text{ and } \tau^{(n)}(t) = \min \{ \tau^y, t, \tau_n \}.
\]
Integrating the above equation from 0 to \( r^{(n)}(t) \) and solving the expectation with the help of Dykins’ formula yields
\[
E\mathcal{Y}\left(S\left(r^{(n)}(t)\right), I\left(r^{(n)}(t)\right), V\left(r^{(n)}(t)\right)\right) - \mathcal{Y}(y) = E\int_0^{r^{(n)}(t)} \mathcal{L}\mathcal{Y}(S(u), I(u), M(u), V(u))du \\
\leq E\int_0^{r^{(n)}(t)} -1du = -E\tau^{(n)}(t).
\]
Since \( \mathcal{Y}(y) \) is non-negative, it is obvious that \( E\tau^{(n)}(t) \leq \mathcal{Y}(y) \). We can get \( P\{\tau_c = \infty\} = 1 \) from the proof of Theorem 1. Thus, the system is regular; that is, for \( t \to \infty, n \to \infty \), we have \( \tau^y(t) \to \tau^y \) almost surely. According to Fatou Lemma, we have \( E\tau^y \leq \mathcal{Y}(y) < \infty \). It is easy to find that for every compact set \( R^4 \setminus D \) of \( R^4 \), \( \sup_{y \in R^4 \setminus D} E\tau^y < \infty \). Then, the condition (ii) in Lemma 2 is satisfied. 

7. Numerical simulations

Numerical simulations are presented to support our theoretical findings of the model (1.2) and reveal the impact of media coverage on the spread of disease. Using the Milstein method mentioned in Higham [46], we consider the discretized equations as follows:

\[
S_{i+1} = S_i + \left( \Lambda_1 - d_1S_i - \left( \frac{\beta_1 - \beta_2I_i}{m + I_i} \right) \frac{S_iI_i}{1 + \alpha_1I_i} - \frac{\beta_3S_iV_i}{1 + \alpha_2V_i} \right) \Delta t \\
+ \left( \sigma_1 \sqrt{\Delta t} \zeta_1 \right) \frac{S_iI_i}{1 + \alpha_1I_i} + \frac{\sigma_1}{2} \left( \frac{\sigma_1^2}{2} \right) \Delta t, \\
I_{i+1} = I_i + \left( \frac{\beta_1 - \beta_2I_i}{m + I_i} \right) \frac{S_iI_i}{1 + \alpha_1I_i} + \frac{\beta_3S_iV_i}{1 + \alpha_2V_i} - (d_1 + \mu + \gamma)I_i \Delta t \\
+ \left( \sigma_2 \sqrt{\Delta t} \zeta_2 \right) \frac{S_iI_i}{1 + \alpha_1I_i} + \frac{\sigma_2^2}{2} \Delta t, \\
M_{i+1} = M_i + \left( \frac{\Lambda_2 - \beta_4M_i}{1 + \alpha_3I_i} - d_2M_i \right) \Delta t + M_i \left( \frac{\sigma_3 \sqrt{\Delta t} \zeta_3}{2} + \frac{\sigma_3^2}{2} M_i \right) \Delta t, \\
V_{i+1} = V_i + \left( \frac{\beta_4M_i}{1 + \alpha_3I_i} - d_2V_i \right) \Delta t + V_i \left( \frac{\sigma_4 \sqrt{\Delta t} \zeta_4}{2} + \frac{\sigma_4^2}{2} V_i \right) \Delta t,
\]

where the time increment \( \Delta t > 0 \), \( \zeta_1, \zeta_2, \zeta_3, \zeta_4 \), are mutually independent Gaussian random variables which follow the distribution \( N(0, 1) \) for \( i = 0, 1, 2, ..., n \).

Vector-borne diseases with two transmission routes may be more likely to become endemic than diseases with one transmission route. Therefore, we tend to choose lower transmission rates and recruitment when numerically modeling disease extinction.

Example 1. Let \( \Lambda_1 = 100, \Lambda_2 = 100, \beta_1 = 0.000012, \beta_2 = 0.0000018, \beta_3 = 0.000039, \beta_4 = 0.000039, \alpha_1 = 0.13, \alpha_2 = 0.15, \alpha_3 = 0.15, \mu = 0.13, \gamma = 0.13, d_1 = 0.1, d_2 = 0.1, m = 20, \sigma_1 = 0.025, \sigma_2 = 0.25, \sigma_3 = 0.03, \sigma_4 = 0.26, \mu_1 = \min(\mu, d_1, \gamma), \mu_2 = \min(\sigma_2, \sigma_4), \beta = \max(\beta_1, \beta_3) \), and the initial values \( (S(0), I(0), M(0), V(0)) = (1000, 15, 1000, 50) \). So
\[
\mathcal{R}_0^S = \frac{1}{\mu_1 + \sigma_2^2/2} \left( \frac{\beta \Lambda_1}{d_1} + \frac{\beta_4 \Lambda_2}{d_2} \right) \approx 0.594 < 1.
\]
According to Theorem 2, the solution of the stochastic model (1.2) will eventually approach zero; this means the disease will die out almost surely. And from Fig. 1, it is observed that the number of infected individuals tends to zero.

Example 2. We keep the parameters the same as in Example 1 except that \( \Lambda_1 = \Lambda_2 = 500 \), \( \beta_1 = 0.01 \) and \( \beta_3 = \beta_4 = 0.001 \). Then

\[
\mathcal{R}_1^S = \frac{9 \sqrt{\Lambda_1^2 \Lambda_2^2 d_1^2 d_2 (\beta_1 - \beta_2) \beta_3 \beta_4} + \frac{1}{2} \left( \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 \right)}{4d_1 + 3d_2 + \mu + \gamma} \approx 3.555 > 1.
\]

Theorem 3 implies that the disease is persistent in the mean. Interestingly, In Fig. 2, it is clear that the number of infected individuals is higher than that of susceptible individuals.
Example 3. Choose parameters $\Lambda_1 = 35000$, $\Lambda_2 = 30000$, $\beta_1 = 0.05$, $\beta_2 = 0.000002$, $\beta_3 = 0.069$, $\beta_4 = 0.069$, $\alpha_1 = 0.1$, $\alpha_2 = 0.12$, $\alpha_3 = 0.12$, $\mu = 0.23$, $\gamma = 0.2$, $d_1 = 0.5$, $d_2 = 0.058$, $m = 100$, $\sigma_1 = 0.015$, $\sigma_2 = 0.018$, $\sigma_3 = 0.018$, $\sigma_4 = 0.02$, and the initial values $\mu_1 = \min(\mu+d_1+\gamma, d_2)$, $\sigma = \min(\sigma_2, \sigma_3)$, $\beta = \max(\beta_1, \beta_3)$, and the initial values $(S(0), I(0), M(0), V(0)) = (20000, 2000, 20000, 2000)$. The parameters

$$R^S_3 = 7 \sqrt{\frac{(\beta_1 - \beta_2)\beta_3\Lambda_1^2\Lambda_2}{4}} \left( \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{\sigma_4^2}{2} + 2d_1 + 3d_2 + (\mu + \gamma) + \frac{\beta_1}{\alpha_1} + \frac{\beta_3}{\alpha_2} + \frac{\beta_4}{\alpha_3} \right),$$

$$R^S_4 = 8 \sqrt{\frac{(\beta_1 - \beta_2)\beta_3\Lambda_1^2\Lambda_2}{4}} \left( \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{\sigma_4^2}{2} + 4d_1 + 2d_2 + (\mu + \gamma) + \frac{\beta_1}{\alpha_1} + \frac{\beta_3}{\alpha_2} + \frac{\beta_4}{\alpha_3} \right),$$

we can obtain that $R^S_2 = \{R^S_3, R^S_4\} \approx 49.720 > 1$ and the conditions of Theorem 4 is satisfied. The Fig. 3 show that the histograms of solutions of model (1.2) with white noise. Theoretical conclusions and numerical simulations indicate that the disease will eventually prevail and persist for a long time.

Example 4. When $\beta_2 = 0$, different transmission rates $\beta_1 = 0, 0.02, 0.05, 0.08, 0.12$, are selected, and when the transmission rate $\beta_1 = 0.05$, $\beta_2 = 0, 0.006, 0.01, 0.012, 0.016, 0.02$ are selected, and the rest of the parameters are the same as in Example 1. As shown in Fig 4. , As $\beta_1$ changes...
from 0, it significantly impacts them of the system. As $\beta_2$ increases, the size of the infections decreases. This shows that the existence of direct transmission of this transmission route has an important impression on disease transmission, and reducing the rate of human-to-human contact through media coverage can reduce the scale of vector-borne infectious diseases.

8. Conclusions

This paper studies a direct transmission model saturated with stochastic vector-borne disease incidence and the associated dynamical behavior. We obtain positive definiteness and uniqueness of the solution to the stochastic model. Then, we establish sufficient conditions for the extinction of the disease in both populations. Furthermore, we prove the uniqueness of the existence of an ergodic stationary distribution of the model when $R_2^S > 1$ by choosing a suitable stochastic Lyapunov function.

On the other hand, from the simulation, we found that the disease under increasing transmission rate $\beta_1$ showed an increasing transmission scale. It reflected that direct transmission, the transmission route, has an essential influence on the spread of vector-borne diseases. In addition, we observed in the numerical experiments that there is indeed an effect on the size of the infection by increasing the value of the $\beta_2$. This also validates the inhibitory impact of media coverage on the spread of the disease.

Finally, reviewing the model we built, we can find that model (1.1) is the classical SIR model with media coverage if we make $\beta_3 = \beta_4 = 0$. This means that the disease can be endemic in the host population if there is no transmission pathway from the vector to the host. The threshold parameters obtained in this paper do not explain this phenomenon. Some algorithms that guarantee the positivity of the solution are more useful when numerical simulations are performed [47, 48]. We leave these issues for future research.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in creating this article.
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Conflict of interest

The authors declare that they have no conflict of interest.

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