Research article

The Hermite-Hadamard Inequality for $s$-Convex Functions in the Fourth Sense

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Abstract: In this paper, some Hermite-Hadamard type inequalities were derived for $s$-convex functions in the fourth sense and some applications connected with special means were given.

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1. Introduction

The theory of convex functions plays an important role in different fields of pure and applied sciences. Over the years, using different properties of convex functions, many researchers introduced abstract convex function types (see [4, 11, 15] and the references therein). Because of their characteristic properties, convex functions satisfy some well-known inequalities such as Jensen, Fejer. One of this kind of inequalities is the Hermite-Hadamard inequality which gives us an estimate of the mean value of a convex function which works great in Analysis. This inequality was thought to be first described by Hadamard in 1893. However, it was studied 10 years ago by C. Hermite [10].

The Hermite-Hadamard inequality is (1.1) for a convex function defined on the interval $[a, b]$,

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}. \quad (1.1)$$

This inequality attracts special interests of many researchers. They have presented various refinements, extensions, generalizations for the convex functions and different types of abstract convex functions (see [1, 2, 6, 7, 12, 18, 19, 23] and the references therein). $s$-convexity, which originates from the studies on modular spaces [5], is one of the abstract convexity type and has some applications, especially, in fractal theory [16].
In this article, we present the Hermite-Hadamard inequality for the $s$-convex functions in the fourth sense and an application is given. In addition, generator functions and some properties of generator functions are given for the class of $s$-convex functions in the fourth sense. Also some integral inequalities are given under different circumstances.

2. Preliminaries

This section provides basic information that we need in the article.

Let $U$ be a subset in $\mathbb{R}^n$. The set $U$ is said to be convex set in $\mathbb{R}^n$ whenever it contains two points, it also contains the line segment joining them. Its algebraic expression is that $U$ is convex set if

$$\lambda x + \mu y \in U$$

whenever $x, y \in U$ and $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$.

Let $U$ be a subset in $\mathbb{R}^n$ and $s \in (0, 1]$. The set $U$ is said to be $s$-convex set in $\mathbb{R}^n$, if

$$\lambda x + \mu y \in U$$

whenever $x, y \in U$ and $\lambda, \mu \geq 0$ with $\lambda^s + \mu^s = 1$ [3]. Also, it is called as $p$-convex set in some articles.

**Definition 1.** Let $U$ be a convex set on vector space $\mathbb{R}^n$. A function $f : U \rightarrow \mathbb{R}$ is called a convex function on $U$, if the following inequality holds,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \quad (2.1)$$

for all $x, y \in U$ and $\lambda \in [0, 1]$.

Many new classes of convex functions have been obtained with $\lambda$ under different conditions. One of them is the $s$-convex function class. The following definitions of the two different sense of $s$-convexity $(0 < s \leq 1)$ of real-valued functions are known in the literature.

**Definition 2.** [17] Let $U$ be a $s$-convex set. A function $f : U \rightarrow \mathbb{R}$ is said to be $s$-convex function in the first sense if the following inequality holds,

$$f(\lambda x + \mu y) \leq \lambda^s f(x) + \mu^s f(y) \quad (2.2)$$

for all $x, y \in U$ and $\lambda, \mu \geq 0$ with $\lambda^s + \mu^s = 1$.

This definition of $s$-convex function for $\phi$-functions was introduced by Orlicz in [17] and used in the theory of Orlicz spaces [14, 20, 21]

**Definition 3.** [5] Let $U$ be a convex set on vector space $\mathbb{R}^n$. A function $f : U \rightarrow \mathbb{R}$ is said to be $s$-convex function in the second sense if the following inequality holds,

$$f(\lambda x + \mu y) \leq \lambda^s f(x) + \mu^s f(y) \quad (2.3)$$

for all $x, y \in U$ and $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$.

In $s$-convexity, it is possible to define new function classes by introducing different conditions on the coefficients [13, 22]. These function classes can be considered as continuation of the $s$-convex function classes in the first and second sense.
Definition 4. [13] Let $U$ be a $s$-convex set. A function $f : U \to \mathbb{R}$ is said to be $s$-convex function in the third sense if the following inequality holds,

$$f(\lambda x + \mu y) \leq \lambda^s f(x) + \mu^s f(y)$$

(2.4)

for all $x, y \in U$ and $\lambda, \mu \geq 0$ with $\lambda^s + \mu^s = 1$, and for some fixed $s \in (0, 1]$.

Definition 5. [22] Let $U$ be a convex set. A function $f : U \to \mathbb{R}$ is said to be $s$-convex function in the fourth sense if the following inequality holds,

$$f(\lambda x + \mu y) \leq \lambda^{\frac{1}{s}} f(x) + \mu^{\frac{1}{s}} f(y)$$

(2.5)

for all $x, y \in U$ and all $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$, and for some fixed $s \in (0, 1]$. This inequality (2.5) is equivalent to the following inequality,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda^{\frac{1}{s}} f(x) + (1 - \lambda)^{\frac{1}{s}} f(y),$$

where $\lambda \in [0, 1]$ and $x, y \in U$.

The class of $s$-convex functions in the fourth sense is denoted by $K_4^s$.

It can be easily seen that for $s = 1$, $s$-convexity is reduced to the ordinary convexity of functions defined on $\mathbb{R}^n$.

Many researchers have worked on many inequalities for different convexity classes. One of the popular of these inequalities is the Hermite-Hadamard inequality (1.1). In this study, Hermite-Hadamard inequality is studied for the class $K_4^s$, which is one of the new convex function classes.

Hermite-Hadamard inequality for $s$-convex functions in the first and second sense are given in the following two theorems, respectively.

Theorem 6. [8] Suppose that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a $s$-convex function in the first sense where $s \in (0, 1]$ and let $a, b \in \mathbb{R}_+$ with $a \leq b$. Then the following the inequality holds,

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + s f(b)}{s + 1}.$$ 

Theorem 7. [9] Suppose that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a $s$-convex function in the second sense where $s \in (0, 1]$ and let $a, b \in \mathbb{R}_+$ with $a \leq b$. Then the following inequality holds,

$$2^{s-1} f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{s + 1}.$$ 

Some results obtained from the properties of $s$-convex function in the fourth sense functions in [22] that we use are follows:

Corollary 8. Let $f : U \to \mathbb{R}$ and $f \in K_4^s$. Then, the following inequality is valid for all $x, y \in U$ :

$$f\left(\frac{x + y}{2}\right) \leq \frac{f(x) + f(y)}{2^{\frac{1}{s}}}.$$ 

(2.6)
Proof. If we take $\lambda = \mu = \frac{1}{2}$ in inequality (2.5), the proof is complete. □

Corollary 9. Let $U$ be a convex set. If $f : U \to \mathbb{R}$ is a $s$-convex function in the fourth sense, then $f \leq 0$.

Proof. If we take $x = y$ in inequality (2.6), then we get

$$f(x) \leq 2^{1-s}f(x).$$

So, $(1 - 2^{1-s})f(x) \leq 0$. From here it is clear that $f(x) \leq 0$. □

Similarly, it is deduced that if $f$ is $s$-concave function in the fourth sense, then $f \geq 0$.

3. Hermite-Hadamard inequality for $s$-convex functions in the fourth sense

In this part, Hermite-Hadamard inequality is obtained for $s$-convex function classes in the fourth sense. In addition, its applications are shown by examples.

Theorem 10. Let $f : [a, b] \to \mathbb{R}$ be $s$-convex function in the fourth sense and integrable on $[a, b]$, then the following inequality holds,

$$2^{1-s}f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_{a}^{b} f(x)dx \leq \frac{s}{s + 1} [f(a) + f(b)]. \quad (3.1)$$

Both the inequalities hold in reversed direction if $f$ is concave on $[a, b]$.

Proof. As $f$ is $s$-convex function in the fourth sense on $[a, b]$, and if we take $x = ta + (1 - t)b$, then we have

$$f(ta + (1 - t)b) \leq t^{\frac{1}{s}} f(a) + (1 - t)^{\frac{1}{s}} f(b)$$

for all $t \in [0, 1]$. Integrating upper inequality on $[0, 1]$, we get

$$\int_{a}^{b} f(x)dx = \int_{0}^{1} f(ta + (1 - t)b)(b - a)dt$$

$$\leq (b - a) \int_{0}^{1} \left( t^{\frac{1}{s}} f(a) + (1 - t)^{\frac{1}{s}} f(b) \right) dt$$

$$= \frac{s}{s + 1} (b - a) (f(a) + f(b)).$$

Thus, we get the second part of (3.1),

$$\frac{1}{b - a} \int_{a}^{b} f(x)dx \leq \frac{s}{s + 1} (f(a) + f(b)). \quad (3.2)$$

From inequality (2.6), we have

$$f\left(\frac{x + y}{2}\right) \leq \frac{f(x) + f(y)}{2^{\frac{1}{s}}}$$
for $x, y \in [a, b]$. If we put $x = ta + (1-t)b$, $y = (1-t)a + tb$, then we get

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{ta + (1-t)b + (1-t)a + tb}{2}\right) \leq \frac{1}{2^{1+s}} (f(ta + (1-t)b) + f((1-t)a + tb)).$$

Integrating this inequality, we have the first part of (3.1),

$$\int_0^1 f\left(\frac{a+b}{2}\right) dt \leq \frac{1}{2^{1+s}} \int_0^1 (f(ta + (1-t)b) + f((1-t)a + tb)) dt \leq \frac{1}{2^{1+s}} \int_0^1 f(ta + (1-t)b) dt \leq \frac{1}{2^{1+s}} \int_a^b f(x) dx,$$

where we use the following equality,

$$\int_0^1 f(ta + (1-t)b) dt = \int_0^1 f((1-t)a + tb) dt.$$

Thus we get

$$2^{1+s-1} f\left(\frac{a+b}{2}\right) dt \leq \frac{1}{b-a} \int_a^b f(x) dx.$$

From inequalities (3.2) and (3.3) we can write the following inequality

$$2^{1+s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{s}{s+1} (f(a) + f(b)).$$

**Remark 11.** In the inequality (3.1), if we choose $s = 1$, this inequality becomes the classical Hermite-Hadamard inequality (1.1).

In the following theorem, a generator function is defined for the class $K^s_4$ and some properties of this generator function are given.

**Theorem 12.** Let $f$ be an integrable function on $[a, b]$ and let $G$ be defined as follows:

$$G(t) := \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx$$

(3.4)
for $t \in [0, 1]$.

i) If $f$ is a $s$-convex function in the fourth sense on interval $[a, b]$, then $G$ is $s$-convex function in the fourth sense on $[0, 1]$.

ii) If $f$ is a $s$-convex function in the fourth sense on interval $[a, b]$, then the following inequality holds,

$$G(t) \geq 2^{\frac{1}{s}-1} \cdot f\left(\frac{a + b}{2}\right).$$ (3.5)

Proof. i) Let $t_1, t_2 \in [0, 1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. By using $f$ is a $s$-convex function in the fourth sense on interval $[a, b]$, we get

$$G(\alpha t_1 + \beta t_2) = \frac{1}{b - a} \int_a^b f\left((\alpha t_1 + \beta t_2)x + (1 - \alpha t_1 - \beta t_2) \frac{a + b}{2}\right) dx$$

$$= \frac{1}{b - a} \int_a^b f\left(\alpha \left(tx + (1 - t) \frac{a + b}{2}\right) + \beta \left(tx + (1 - t) \frac{a + b}{2}\right)\right) dx$$

$$\leq \frac{1}{b - a} \int_a^b \left[\alpha^\frac{1}{s} f\left(tx + (1 - t) \frac{a + b}{2}\right) + \beta^\frac{1}{s} f\left(tx + (1 - t) \frac{a + b}{2}\right)\right] dx$$

$$= \alpha^\frac{1}{s} \frac{1}{b - a} \int_a^b f\left(tx + (1 - t) \frac{a + b}{2}\right) dx + \beta^\frac{1}{s} \frac{1}{b - a} \int_a^b f\left(tx + (1 - t) \frac{a + b}{2}\right) dx$$

$$= \alpha^\frac{1}{s} G(t_1) + \beta^\frac{1}{s} G(t_2).$$

It shows that $G$ is $s$-convex function in the fourth sense on $[a, b]$.

ii) In the case of $t \in (0, 1]$, the following equality is obtained by taking $u = tx + (1 - t) \frac{a + b}{2}$

$$G(t) = \frac{1}{p - q} \int_q^p f(u) du$$

where $p = tb + (1 - t) \frac{a + b}{2}$ and $q = ta + (1 - t) \frac{a + b}{2}$.

Applying inequality (3.1), we have

$$G(t) = \frac{1}{p - q} \int_q^p f(u) du$$

$$\geq 2^{\frac{1}{s}-1} f\left(\frac{p + q}{2}\right)$$

$$= 2^{\frac{1}{s}-1} f\left(\frac{a + b}{2}\right)$$

and the inequality (3.5) is obtained.

In case of $t = 0$, since $f\left(\frac{a + b}{2}\right) < 0$ and $2^{\frac{1}{s}-1} \geq 1$, the inequality (3.5) is also provided, i.e.;

$$G(0) = f\left(\frac{a + b}{2}\right) \geq 2^{\frac{1}{s}-1} f\left(\frac{a + b}{2}\right).$$
Theorem 13. Let the functions $G_1(t)$ and $G_2(t)$ be defined as follows:

$$G_1(t) := \frac{1}{b-a} t^{\frac{1}{2}} \int_a^b f(x)dx + (1-t)^{\frac{1}{2}} f \left( \frac{a+b}{2} \right)$$

and

$$G_2(t) := \frac{s}{s+1} \left( f \left( ta + (1-t) \frac{a+b}{2} \right) + f \left( tb + (1-t) \frac{a+b}{2} \right) \right)$$

for $t \in [0,1]$. If $f$ is $s$-convex function in the fourth sense on $[a,b]$, then

$$G(t) \leq \min (G_1(t), G_2(t)) \quad (3.6)$$

for $t \in [0,1]$, where $G(t)$ is defined as in Theorem 12.

Proof. In the case of $t \in (0,1]$, we have the following inequality,

$$G(t) = \frac{1}{p-q} \int_q^p f(u)du$$

$$\leq \frac{s}{1+s} \left( f(p) + f(q) \right)$$

$$= \frac{s}{1+s} \left( f(tb + (1-t) \frac{a+b}{2}) + f(ta + (1-t) \frac{a+b}{2}) \right).$$

So, $G(t) \leq G_2(t)$.

For $G_1(t)$, since $f$ is $s$-convex function in the fourth sense, we have

$$f \left( tx + (1-t) \frac{a+b}{2} \right) \leq t^{\frac{1}{2}} f(x) + (1-t)^{\frac{1}{2}} f \left( \frac{a+b}{2} \right)$$

and integrating this inequality on $[a,b]$ we get

$$G(t) \leq \frac{1}{b-a} t^{\frac{1}{2}} \int_a^b f(x)dx + (1-t)^{\frac{1}{2}} f \left( \frac{a+b}{2} \right).$$

So, $G(t) \leq G_1(t)$.

In case of $t = 0$,

$$G(0) = \frac{1}{b-a} \int_a^b f \left( \frac{a+b}{2} \right) dx = f \left( \frac{a+b}{2} \right)$$

so, $G(0) = G_1(0)$ and $G_2(0) = \frac{2s}{s+1} f \left( \frac{a+b}{2} \right)$. Since $f \left( \frac{a+b}{2} \right) < 0$ and $0 < \frac{2s}{s+1} < 1$, we get

$$G(0) = f \left( \frac{a+b}{2} \right) \leq \frac{2s}{s+1} f \left( \frac{a+b}{2} \right).$$

Thus, $G(t) \leq G_2(t)$. □
Theorem 14. Let \( f \) be a \( s \)-convex function in the fourth sense on \([a, b]\) and let \( G_1(t), G_2(t) \) be defined as in Theorem 13. If \( \widetilde{G} := \max (G_1(t), G_2(t)) \) for \( t \in [0, 1] \), then

\[
\widetilde{G}(t) \leq \frac{s}{s+1} \left( t^\frac{s}{2} (f(a) + f(b)) + (1-t)^{\frac{s}{2}} 2f \left( \frac{a+b}{2} \right) \right).
\]

Proof. By inequality (3.1) and since \( 0 < \frac{2s}{s+1} < 1 \), we can write the following inequalities,

\[
\frac{1}{b-a} t^\frac{s}{2} \int_a^b f(x) dx \leq \frac{s}{s+1} t^\frac{s}{2} (f(a) + f(b)),
\]

and

\[
(1-t)^\frac{s}{2} f \left( \frac{a+b}{2} \right) \leq (1-t)^\frac{s}{2} \frac{2s}{s+1} f \left( \frac{a+b}{2} \right)
\]

for \( t \in (0, 1) \). So that

\[
G_1(t) = \frac{1}{b-a} t^\frac{s}{2} \int_a^b f(x) dx + (1-t)^\frac{s}{2} f \left( \frac{a+b}{2} \right)
\]

\[
\leq t^\frac{s}{2} \frac{s}{s+1} (f(a) + f(b)) + (1-t)^\frac{s}{2} \frac{2s}{s+1} f \left( \frac{a+b}{2} \right)
\]

\[
= \frac{s}{s+1} \left[ t^\frac{s}{2} (f(a) + f(b)) + (1-t)^\frac{s}{2} 2f \left( \frac{a+b}{2} \right) \right]
\]

On the other hand, we have

\[
G_2(t) = \frac{s}{s+1} \left[ f \left( ta + (1-t) \frac{a+b}{2} \right) + f \left( tb + (1-t) \frac{a+b}{2} \right) \right]
\]

\[
\leq \frac{s}{s+1} \left[ t^\frac{s}{2} f(a) + (1-t)^\frac{s}{2} f \left( \frac{a+b}{2} \right) + t^\frac{s}{2} f(b) + (1-t)^\frac{s}{2} f \left( \frac{a+b}{2} \right) \right]
\]

\[
= \frac{s}{s+1} \left[ t^\frac{s}{2} (f(a) + f(b)) + (1-t)^\frac{s}{2} 2f \left( \frac{a+b}{2} \right) \right].
\]

Thus, we get \( \widetilde{G} \leq \frac{2s}{s+1} \left( t^\frac{s}{2} (f(a) + f(b)) + (1-t)^\frac{s}{2} 2f \left( \frac{a+b}{2} \right) \right) \). \( \square \)

In the following theorem, a new generator function for the class \( K^4_4 \) is given by a double integrable function which is \( s \)-convex function in the fourth sense and the properties of the new generator function are mentioned.

Theorem 15. Let \( f \) be integrable function on \([a, b]\) and

\[
F(t) := \frac{1}{(b-a)^2} \int_a^b \int_a^b f \left( tx + (1-t)y \right) dx dy
\]

for \( t \in [0, 1] \). If \( f \) is \( s \)-convex function in the fourth sense on \([a, b]\), then \( F \) is also \( s \)-convex function in the fourth sense.
Proof. Let \( t_1, t_2 \in [0, 1] \) and \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \). Then

\[
F(\alpha t_1 + \beta t_2) = \frac{1}{(b-a)^2} \int_a^b \int_a^b f((\alpha t_1 + \beta t_2)x + (1 - \alpha t_1 - \beta t_2)y) \, dxdy
\]

\[
= \frac{1}{(b-a)^2} \int_a^b \int_a^b f(\alpha t_1 x + (1 - t_1)y) + \beta t_2 x + (1 - t_2)y) \, dxdy
\]

\[
\leq \alpha^\frac{1}{2} \frac{1}{(b-a)^2} \int_a^b \int_a^b f(t_1 x + (1 - t_1)y) \, dxdy + \beta^\frac{1}{2} \frac{1}{(b-a)^2} \int_a^b \int_a^b f(t_2 x + (1 - t_2)y) \, dxdy
\]

\[
= \alpha^\frac{1}{2} F(t_1) + \beta^\frac{1}{2} F(t_2).
\]

\[\square\]

**Theorem 16.** Let \( f \) be \( s \)-convex function in the fourth sense on \([a, b]\). Then for \( t \in [0, 1] \), we have

i) \( 2^{\frac{1}{2}} F(t) \geq \frac{1}{(b-a)} \int_a^b \int_a^b f(\frac{x+y}{2}) \, dxdy \),

ii) \( F(t) \geq 2^{\frac{1}{2}} - 1 \max \{G(t), G(1-t)\} \),

iii) \( F(t) = \left(t^2 + (1-t)^2\right) \frac{1}{b-a} \int_a^b f(x)dx\),

iv) \( F(t) \leq \frac{x^2}{(x+1)^2} \left[f(a) + f(ta + (1-t)b) + f(tb + (1-t)a) + f(b)\right] \).

**Proof.**

i) From \( s \)-convexity in the fourth sense of \( f \), we get

\[
f(\frac{x+y}{2}) \leq \frac{f(tx + (1-t)y) + f(ty + (1-t)x)}{2^{\frac{1}{2}}},
\]

for all \( t \in [0, 1] \) and \( x, y \in [a, b] \). Integrating this inequality on \([a, b]^2\), we get

\[
\int_a^b \int_a^b f(\frac{x+y}{2}) \, dxdy \leq \int_a^b \int_a^b \frac{f(tx + (1-t)y) + f(ty + (1-t)x)}{2^{\frac{1}{2}}} \, dxdy
\]

here

\[
\int_a^b \int_a^b f(tx + (1-t)y) \, dxdy = \int_a^b \int_a^b f(ty + (1-t)x) \, dxdy
\]

this yields (i).

ii) For \( y \in [a, b] \), let us define a function \( G_y(t) \) as follows,

\[
G_y(t) := \frac{1}{b-a} \int_a^b f(tx + (1-t)y)dx.
\]
In case of \( t \in (0, 1] \) we take \( u = tx + (1 - t)y \), then

\[
G_y(t) = \frac{1}{p - q} \int_q^p f(u)du
\]

where \( p = tb + (1 - t)y \) and \( q = ta + (1 - t)y \). Applying inequality (3.1), we get

\[
\frac{1}{p - q} \int_q^p f(v) dv \geq 2^{\frac{1}{s} - 1} f\left(\frac{p + q}{2}\right) = 2^{\frac{1}{s} - 1} f\left(\frac{a + b}{2}\right) + (1 - t)y
\]

and integrating this inequality on \([a, b]\) with respect to \( y \) we find, \( F(t) \geq 2^{\frac{1}{s} - 1} G(1 - t) \). Since \( F(t) = F(1 - t) \) we get (ii).

In case of \( t = 0 \) we have

\[
F(0) = \frac{1}{b - a} \int_a^b f(y)dy
\]

\[
G(0) = f\left(\frac{a + b}{2}\right)
\]

and

\[
f\left(\frac{a + b}{2}\right) 2^{s - 1} \geq f\left(\frac{a + b}{2}\right) 2^{\frac{1}{s} - 1}.
\]

Applying inequality (3.1), we get

\[
2^{\frac{1}{s} - 1} f\left(\frac{a + b}{2}\right) = 2^{\frac{1}{s} - 1} G(0) \leq \frac{1}{b - a} \int_a^b f(x)dx = F(0).
\]

That is,

\[
F(0) \geq 2^{\frac{1}{s} - 1} G(0). \quad (3.7)
\]

\[
2^{\frac{1}{s} - 1} G(1) = 2^{\frac{1}{s} - 1} \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{1}{b - a} \int_a^b f(x)dx = G(1) = F(0).
\]

So,

\[
F(0) \geq 2^{\frac{1}{s} - 1} G(1). \quad (3.8)
\]

From inequalities (3.7) and (3.8), we get \( F(0) \geq 2^{\frac{1}{s} - 1} \max(G(0), G(1)) \).

(iii) If we integrate the following inequality over \([a, b]^2\),

\[
f(tx + (1 - t)y) \leq t^{\frac{1}{s}} f(x) + (1 - t)^{\frac{1}{s}} f(y),
\]

we have

\[
\frac{1}{(b - a)^2} \int_a^b \int_a^b f(tx + (1 - t)y) dxdy \leq \left( t^{\frac{1}{s}} + (1 - t)^{\frac{1}{s}} \right) \frac{1}{b - a} \int_a^b f(x)dx.
\]
That is
\[ F(t) \leq (t^2 + (1 - t)^2) \frac{1}{b-a} \int_a^b f(x)dx. \]

iv) Now observe that, in the notation above, we have
\[ G_s(t) = \frac{1}{p-q} \int_q^p f(u)du \leq \frac{s}{s+1} \frac{(f(ta + (1-t)y) + f(tb + (1-t)y))}{b-a} \]
so that integrating this inequality on \([a, b]\), we get
\[
\frac{1}{b-a} \int_a^b \left( \frac{1}{p-q} \int_q^p f(u)du \right) dy \leq \int_a^b \frac{s}{s+1} \frac{(f(ta + (1-t)y) + f(tb + (1-t)y))}{b-a} dy \\
\leq \frac{s}{(s+1)(b-a)} \left[ \int_a^b f(ta + (1-t)y)dy + \int_a^b f(tb + (1-t)y)dy \right].
\]

As above we have
\[ G_s(t) \leq \frac{s}{s+1} (f(ta + (1-t)y) + f(tb + (1-t)y)) \]
for \(y \in [a, b]\). From this inequality, we have the following inequalities:
\[ G_a(1-t) \leq \frac{s}{s+1} (f(a) + f(ta + (1-t)b)), \]
\[ G_b(1-t) \leq \frac{s}{s+1} (f(b) + f(tb + (1-t)a)) \]
and we adding the above two inequalities are given the (iv).

\[ \square \]

4. Applications

We consider the applications of Theorem 10 to get some inequalities related to the special means. Let us recall the following means for arbitrary real numbers \(a, b\).

Let \(a, b, p\) be positive number with \(a \neq b\) and \(p \neq 1\),
\[
A(a, b) = \frac{a+b}{2}, \quad L_p(a, b) = \begin{cases} a, & \text{if } a = b \\ \left(\frac{a^p - b^p}{p(a-b)}\right)^{1/(p-1)}, & \text{else} \end{cases}, \quad P_p(a, b) = \left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}}
\]
are called Arithmetic mean, Stolarsky mean (Generalized Logarithmic mean) and Power mean, respectively.
Let \(0 < s < 1\) and \(a, b, c \in \mathbb{R}\) with \(b < 0, c \leq 0\).

\[
f(x) = \begin{cases} 
  a, & \text{if } x = 0 \\
  bx^{\frac{1}{s}} + c, & \text{if } x > 0 
\end{cases}
\]  

(4.1)
is s-convex function in the fourth sense on \((0, \infty)\). Under the extra condition \(a = c\), \(f\) is s-convex function in the fourth sense on \([0, \infty)\).

Let us first show that it is s-convex function in the fourth sense. Assume that \(x, y \in (0, \infty)\). Then \(\lambda x + \mu y > 0\) with \(\lambda + \mu = 1\).

\[
f(\lambda x + \mu y) = b(\lambda x + \mu y)^{\frac{1}{s}} + c
\]

\[
\leq b \left( \lambda \frac{1}{s} x^{\frac{1}{s}} + \mu \frac{1}{s} y^{\frac{1}{s}} \right)^{\frac{1}{s}} + c
\]

\[
= b \left( \lambda \frac{1}{s} x^{\frac{1}{s}} + \mu \frac{1}{s} y^{\frac{1}{s}} \right)^{\frac{1}{s}} + c (\lambda + \mu)
\]

\[
\leq b \left( \lambda \frac{1}{s} x^{\frac{1}{s}} + \mu \frac{1}{s} y^{\frac{1}{s}} \right)^{\frac{1}{s}} + c \left( \lambda \frac{1}{s} + \mu \frac{1}{s} \right)
\]

\[
= \lambda \frac{1}{s} \left( bx^{\frac{1}{s}} + c \right) + \mu \frac{1}{s} \left( by^{\frac{1}{s}} + c \right)
\]

\[
= \lambda \frac{1}{s} f(x) + \mu \frac{1}{s} f(y).
\]

For \(x, y \in [0, \infty)\), we have to check only the cases \(x > y = 0\) and \(x = y = 0\).

Let \(y > x = 0\). Then

\[
f(\lambda 0 + \mu y) = f(\mu y) = b\mu \frac{1}{s} y^{\frac{1}{s}} + c = b\mu \frac{1}{s} y^{\frac{1}{s}} + c (\lambda + \mu)
\]

\[
\leq b\mu \frac{1}{s} y^{\frac{1}{s}} + c \left( \lambda \frac{1}{s} + \mu \frac{1}{s} \right) = \lambda \frac{1}{s} c + \mu \frac{1}{s} \left( by^{\frac{1}{s}} + c \right)
\]

\[
= \lambda \frac{1}{s} c + \mu \frac{1}{s} f(y) = \lambda \frac{1}{s} a + \mu \frac{1}{s} f(y) = \lambda \frac{1}{s} f(0) + \mu \frac{1}{s} f(0).
\]

Let \(y = x = 0\). Then

\[
f(\lambda 0 + \mu 0) = a \leq a (\lambda \frac{1}{s} + \mu \frac{1}{s}) = \lambda \frac{1}{s} f(0) + \mu \frac{1}{s} f(0).
\]

Now, using inequality (3.1), we give some inequalities for special means of real numbers.

**Proposition 17.** Let \(a, b \in \mathbb{R}_+\) with \(a < b\). The inequality holds,

\[
2^{1-s} A(a, b) \geq L_{\frac{1}{s}}(a, b) \geq \left( \frac{2s}{s+1} \right)^{s} P_{\frac{1}{s}}(a, b)
\]

(4.2)

for all \(s \in (0, 1]\).

**Proof.** The assertion follows from Theorem 10 applied to the s-convex function in the fourth sense \(f(x) = -x^{\frac{1}{s}}, x \in [a, b]\)

\[
\left( \frac{a + b} {2} \right)^{s} \geq \frac{s}{b-a} \left( b^{\frac{1}{s}+1} - a^{\frac{1}{s}+1} \right) \geq \frac{s}{s+1} \left( a^{\frac{1}{s}} + b^{\frac{1}{s}} \right). 
\]

The \(s\)-th power of each side of the above inequality is

\[
\frac{a + b} {2^s} \geq \left( \frac{1}{s+1} \right)^{s} \left( b^{\frac{1}{s}+1} - a^{\frac{1}{s}+1} \right) \geq \left( \frac{s}{s+1} \right)^{s} \left( a^{\frac{1}{s}} + b^{\frac{1}{s}} \right)^{s}. 
\]
We can write this inequality as follows

\[
2^{1-s}{a + b \over 2} \geq \left[ {b^{s+1} - a^{s+1} \over (s + 1)(b - a)} \right]^s \geq \left( {2s \over s + 1} \right)^s \left( {a^{s+1} + b^{s+1} \over 2} \right)^s.
\] (4.3)

From this inequality, we get

\[
2^{1-s}A(a, b) \geq L^{1-s}_t(a, b) \geq \left( {2s \over s + 1} \right)^s P^{1-s}_t(a, b).
\]

□

Proposition 18. Let \(a, b \in \mathbb{R}^+\), with \(a < b\) and \(s \in (0, 1]\). For \(t \in [0, 1]\), the following holds

\[
\left[ L^{1-s}_t (ta + (1-t)^{a+b \over 2}, tb + (1-t)^{a+b \over 2}) \right]^{1 \over t} \leq 2^{1-1}A^{1 \over t}(a, b).
\] (4.4)

Proof. Let \(f(x) = -x^{1 \over s}\) with \(s \in (0, 1]\) on \([a, b]\). Applying (i) in Theorem 12, we have

\[
G(t) = -s + t \left[ (tb + (1-t)^{a+b \over 2})^{1 \over s+1} - (ta + (1-t)^{a+b \over 2})^{1 \over s+1} \right]
\]

for \(t \in (0, 1]\). Since \(\lim\limits_{t \to 0} G(t) = -\left( {a+b \over 2} \right)^{1 \over s}\), we can consider \(G(0) = -\left( {a+b \over 2} \right)^{1 \over s}\). Expressing \(G(t)\) as generalized logarithmic mean and applying (ii) in Theorem 12, we have the inequality (4.4) for \(t \in (0, 1]\). For \(t = 0\), from the definition of generalized logarithmic mean,

\[
\left[ L^{1-s}_t \left( {a+b \over 2}, {a+b \over 2} \right) \right]^{1 \over t} = \left( {a+b \over 2} \right)^{1 \over s} \leq 2^{1-1}A^{1 \over t}(a, b)
\]

is shown. Thus, the inequality (4.4) holds for all \(t \in [0, 1]\). □

For \(t = 1\) in Proposition 18, the following inequality is obtained:

Corollary 19. Let \(a, b \in \mathbb{R}^+\), with \(a < b\) and \(s \in (0, 1]\). Then

\[
\left[ L^{1-s}_1(a, b) \right]^{1 \over 1} \leq 2^{1-1}A^{1 \over 1}(a, b).
\]

Also, the obtained results can be used to get an lower and upper bounds specific functions.

Proposition 20. For \(x \geq 3\),

\[
{x \over x - 1} - \gamma \leq \Psi(x) \leq 2^{x-2} - {1 \over 2} - \gamma
\]

where \(\Psi(x)\) is digamma function, i.e.

\[
\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \text{ for } x > 0
\]

and \(\gamma\) is Euler-Mascheroni constant, i.e. \(\gamma \approx 0.5772156649\ldots\)
Proof. Let \( t \in [0, 1] \), \( a = t \) and \( b = 1 \), \( f(x) = -x^{\frac{1}{s}} \) with \( s \in (0, 1] \). Applying Theorem 10, we have

\[
\frac{s}{s+1} (1 + t^{\frac{1}{s}}) \leq \frac{s}{s+1} \frac{1 - t^{\frac{1}{s}+1}}{1 - t} \leq \frac{1}{2} (1 + t)^{\frac{1}{s}}.
\]

By integrating the expression with respect to \( t \) on \([0, 1]\), then,

\[
\frac{2s + 1}{s + 1} \leq \int_0^1 \frac{1 - t^{\frac{1}{s}+1}}{1 - t} dt \leq \frac{2^{\frac{1}{s}+1} - 1}{2}.
\]

Using the following integral representation of digamma function

\[
\Psi(r) = \int_0^1 \frac{1 - t^{-1}}{1 - t} dt - \gamma
\]

for \( r > 0 \), we have

\[
\frac{2s + 1}{s + 1} \leq \Psi\left(\frac{1}{s} + 2\right) + \gamma \leq \frac{2^{\frac{1}{s}+1} - 1}{2}.
\]

The substitution \( x - 2 = \frac{1}{s} \) yields to the desired result.

\( \square \)

Conflict of interest

The authors declare no conflict of interest.

References


22. Z. Eker, S. Sezer, G. Tinaztepe, G. Adilov, \( s \)-convex function in the fourth sense and their some properties. Submitted